

A renormalisation group method.

I. Gaussian integration and normed algebras

David C. Brydges* and Gordon Slade*

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Abstract

This paper is the first in a series devoted to the development of a rigorous renormalisation group method for lattice field theories involving boson fields, fermion fields, or both. Our immediate motivation is a specific model, involving both boson and fermion fields, which arises as a representation of the continuous-time weakly self-avoiding walk. In this paper, we define normed algebras suitable for a renormalisation group analysis, and develop methods for performing analysis on these algebras. We also develop the theory of Gaussian integration on these normed algebras, and prove estimates for Gaussian integrals. The concepts and results developed here provide a foundation for the continuation of the method presented in subsequent papers in the series.

1 Introduction

This paper is the first in a series devoted to the development of a rigorous renormalisation group method. We develop the method with the specific goal of providing the necessary ingredients for our analysis of the critical behaviour of the continuous-time weakly self-avoiding walk in dimension 4 [1, 2], via its representation as a supersymmetric field theory involving both boson and fermion fields [7]. However, our approach is more general, and could potentially be applied also in other settings, including purely bosonic or purely fermionic field theories. Other approaches to the rigorous renormalisation group are discussed in [4].

In the renormalisation group approach, we are interested in performing a Gaussian integral with respect to a positive-definite covariance operator C . The integration is performed progressively: the covariance is decomposed as a sum of positive-definite terms $C = C_1 + C'$ and the original integral is equal to a convolution of Gaussian integrals with respect to C_1 and C' . A proof that decomposition of the covariance corresponds to convolution of Gaussian integrals can be found for our context in [6], but we will give a self-contained proof here within our current formalism and notation.

In order to perform analysis with Gaussian integrals, it is necessary to define suitable norms. In this paper, we define an algebra \mathcal{N} and the T_ϕ *semi-norm* on \mathcal{N} , and prove that the T_ϕ semi-norm

*Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2. E-mail: db5d@math.ubc.ca, slade@math.ubc.ca.

obeys an essential product property. We prove several estimates for the T_ϕ semi-norm, which will be essential for our renormalisation group method, including estimates for Gaussian integrals. In addition, as an example of use of the T_ϕ semi-norm, and as preparation for more detailed estimates obtained in [8], we prove a preliminary estimate for the self-avoiding walk interaction.

The concepts and results from this paper that are needed in subsequent papers in the series are summarised in Section 2, which pertains to Gaussian integration, and in Section 3, which pertains to norms and norm estimates. Most of the proofs are deferred to later sections.

2 Gaussian integration

2.1 Fields and the algebra \mathcal{N}

Given a finite set Λ , and $p \in \mathbb{N}$, let Λ^p denote the p -fold cartesian product of Λ with itself, so that elements of Λ^p are sequences of elements of Λ of length p . We define $\Lambda^0 = \{\emptyset\}$ to be the set whose element is the empty sequence. Then $\Lambda^* = \sqcup_{p=0}^\infty \Lambda^p$ is the set of arbitrary finite sequences of elements of Λ , of any length, including zero. We typically denote the length of $z \in \Lambda^*$ as $p = p(z)$ or $q = q(z)$, and, for $z \in \Lambda^*$, we write $z! = p(z)!$. For $z', z'' \in \Lambda^*$ we define the *concatenation* $z' \circ z''$ to be the sequence in Λ^* whose elements are the elements of z' followed by the elements of z'' .

Let Λ_b be any finite set. An element of \mathbb{R}^{Λ_b} is called a *boson field*, and can be written as $\phi = (\phi_x, x \in \Lambda_b)$. Let $\mathcal{R} = \mathcal{R}(\Lambda_b)$ denote the ring of smooth functions from \mathbb{R}^{Λ_b} to \mathbb{C} . Here *smooth* means having at least $p_{\mathcal{N}}$ continuous derivatives, where $p_{\mathcal{N}}$ is a parameter at our disposal.

Let Λ_f be a finite set and let $\Lambda = \Lambda_b \sqcup \Lambda_f$. The *fermion field* $\psi = (\psi_y, y \in \Lambda_f)$ is a set of anticommuting generators for an algebra $\mathcal{N} = \mathcal{N}(\Lambda)$ over the ring \mathcal{R} . In particular, $\psi_y^2 = 0$ for all $y \in \Lambda_f$. By definition, \mathcal{N} consists of elements F of the form

$$F = \sum_{y \in \Lambda_f^*} \frac{1}{y!} F_y \psi^y, \quad (2.1)$$

where each coefficient F_y is an element of \mathcal{R} , and

$$\psi^y = \begin{cases} 1 & \text{if } q(y) = 0 \\ \psi_{y_1} \cdots \psi_{y_q} & \text{if } q \geq 1 \text{ and } y = (y_1, \dots, y_q). \end{cases} \quad (2.2)$$

We always require F_y to be antisymmetric under permutation of the components of y ; this ensures that the representation (2.1) is unique. We denote F_y evaluated at ϕ by $F_y(\phi)$, and write $F(\phi) = \sum_{y \in \Lambda_f^*} \frac{1}{y!} F_y(\phi) \psi^y$.

Definition 2.1. For A a nonnegative integer, we say that $F \in \mathcal{N}$ is a *polynomial of degree A* if there are coefficients $F_{x,y} \in \mathbb{C}$ such that $F(\phi) = \sum_{x,y: p(x)+q(y) \leq A} \frac{1}{x!y!} F_{x,y} \phi^x \psi^y$, with $F_{x,y} \neq 0$ for some x, y with $p(x) + q(y) = A$.

Polynomial elements of \mathcal{N} play an important role in our analysis. An example of a polynomial of degree 2 is $\phi_w \phi_x + \psi_y \psi_z$, for some $w, x \in \Lambda_b$ and $y, z \in \Lambda_f$.

2.2 Fermionic Gaussian integration

Let Λ'_b and Λ'_f be sets, with an order specified on the elements of Λ'_f . We integrate over fields labelled by elements of these sets, starting in this section with the fermion fields labelled by Λ'_f , and then in Section 2.3 with the boson fields with labels in Λ'_b .

We define the monomial $\psi^{\Lambda'_f}$ to be the product of the generators in the specified order. Let $\Lambda' = \Lambda'_b \sqcup \Lambda'_f$. We write $F \in \mathcal{N}(\Lambda \sqcup \Lambda'_b)$ for the algebra \mathcal{N} with fermion fields indexed by Λ_f and boson fields indexed by $\Lambda_b \sqcup \Lambda'_b$, and $F \in \mathcal{N}(\Lambda \sqcup \Lambda')$ for the algebra \mathcal{N} with fermion fields indexed by $\Lambda_f \sqcup \Lambda'_f$ and boson fields indexed by $\Lambda_b \sqcup \Lambda'_b$.

Definition 2.2. The *Grassmann integral* is the linear map $\int_{\Lambda'_f} : \mathcal{N}(\Lambda \sqcup \Lambda') \rightarrow \mathcal{N}(\Lambda \sqcup \Lambda'_b)$ uniquely defined by the conditions:

- (a) for all $F \in \mathcal{N}(\Lambda \sqcup \Lambda'_b)$, $\int_{\Lambda'_f} F \psi^{y'} = 0$ whenever the elements of $y' \in (\Lambda'_f)^*$ do not form an enumeration of Λ'_f , and
- (b) $\int_{\Lambda'_f} F \psi^{\Lambda'_f} = F$ for all $F \in \mathcal{N}(\Lambda \sqcup \Lambda'_b)$.

The classic reference for Grassmann integration is [3]; accessible and more modern treatments can be found in [11, 13, 16].

Given an antisymmetric invertible $\Lambda'_f \times \Lambda'_f$ matrix \mathbf{A}_f , let

$$S_f = \frac{1}{2} \sum_{u,v \in \Lambda'_f} \mathbf{A}_{f;u,v} \psi_u \psi_v. \quad (2.3)$$

Since the generators anti-commute and since Λ'_f is finite, the series $\sum_{n=0}^{\infty} \frac{1}{n!} (-S_f)^n$ terminates after finitely many terms, and therefore defines an element of $\mathcal{N}(\Lambda \sqcup \Lambda')$. We denote this element by e^{-S_f} . Let \mathbf{C}_f be the inverse of \mathbf{A}_f . The Grassmann analogue of Gaussian integration is the linear map $\mathbb{E}_{\mathbf{C}_f} : \mathcal{N}(\Lambda \sqcup \Lambda') \rightarrow \mathcal{N}(\Lambda \sqcup \Lambda'_b)$ defined by

$$\mathbb{E}_{\mathbf{C}_f} F = N_f \int_{\Lambda'_f} e^{-S_f} F, \quad F \in \mathcal{N}(\Lambda \sqcup \Lambda'), \quad (2.4)$$

where N_f is a normalisation constant such that $\mathbb{E}_{\mathbf{C}_f} 1 = 1$. It is a consequence of [3, (3.16)] that

$$N_f = (\det \mathbf{C}_f)^{1/2}. \quad (2.5)$$

The choice of square root depends on the order we have chosen for Λ'_f . We will be specific below in a less general setting.

2.3 Bosonic Gaussian integration

Given a real symmetric positive-definite $\Lambda'_b \times \Lambda'_b$ matrix \mathbf{A}_b , and given $\phi \in \mathbb{R}^{\Lambda'_b}$, let

$$S_b = \frac{1}{2} \sum_{u,v \in \Lambda'_b} \mathbf{A}_{b;u,v} \phi_u \phi_v. \quad (2.6)$$

The matrix \mathbf{A}_b has positive eigenvalues and therefore an inverse matrix \mathbf{C}_b exists. The Gaussian expectation $\mathbb{E}_{\mathbf{C}_b} : \mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}'_b) \rightarrow \mathcal{N}(\mathbf{\Lambda})$ is the linear map defined as follows. Let $D\phi$ be Lebesgue measure on $\mathbb{R}^{\mathbf{\Lambda}'_b}$. For $F \in \mathcal{R}(\mathbf{\Lambda}_b \sqcup \mathbf{\Lambda}'_b)$, we define

$$\mathbb{E}_{\mathbf{C}_b} F = N_b \int_{\mathbb{R}^{\mathbf{\Lambda}'_b}} e^{-S_b} F D\phi, \quad (2.7)$$

where N_b is chosen so that $\mathbb{E}_{\mathbf{C}_b} 1 = 1$. It is a standard fact about Gaussian integrals that N_b is given by the positive square root

$$N_b = (\det(2\pi\mathbf{C}_b))^{-1/2}. \quad (2.8)$$

Of course $\mathbb{E}_{\mathbf{C}_b}$ is only defined on elements of $\mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}'_b)$ which are such that the growth of the coefficients at infinity is not too rapid. For $F = \sum_{y \in \mathbf{\Lambda}_f} \frac{1}{y!} F_y \psi^y \in \mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}'_b)$, we define

$$\mathbb{E}_{\mathbf{C}_b} F = \sum_{y \in \mathbf{\Lambda}_f} \frac{1}{y!} (\mathbb{E}_{\mathbf{C}_b} F_y) \psi^y. \quad (2.9)$$

2.4 Combined bosonic-fermionic Gaussian integration on \mathcal{N}

Let \mathbf{C} denote the pair $\mathbf{C}_b, \mathbf{C}_f$. Given matrices \mathbf{A}_f and \mathbf{A}_b as above, we define the *combined bosonic-fermionic expectation* to be the linear map $\mathbb{E}_{\mathbf{C}} : \mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}') \rightarrow \mathcal{N}(\mathbf{\Lambda})$ given by

$$\mathbb{E}_{\mathbf{C}} = \mathbb{E}_{\mathbf{C}_b} \mathbb{E}_{\mathbf{C}_f}, \quad (2.10)$$

where $\mathbb{E}_{\mathbf{C}_b}$ acts only on bosons, and $\mathbb{E}_{\mathbf{C}_f}$ acts only on fermions. By linearity, the action of $\mathbb{E}_{\mathbf{C}}$ is determined by its action on KF where $K \in \mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}'_b)$ and F is a monomial in the generators indexed by $\mathbf{\Lambda}'_f$. The map $\mathbb{E}_{\mathbf{C}}$ is defined, for such K, F , by

$$\mathbb{E}_{\mathbf{C}} KF = (\mathbb{E}_{\mathbf{C}_b} K)(\mathbb{E}_{\mathbf{C}_f} F) = \left(N_b \int_{\mathbb{R}^{\mathbf{\Lambda}'_b}} e^{-S_b} K D\phi \right) \left(N_f \int_{\mathbf{\Lambda}'_f} e^{-S_f} F \right). \quad (2.11)$$

On the right-hand side, the boson and fermion fields corresponding to $\mathbf{\Lambda}'$ have been integrated out, leaving dependence only on the fields corresponding to $\mathbf{\Lambda}$.

2.5 The Laplacian

It is ordinary calculus to differentiate a function $f \in \mathcal{R}(\mathbf{\Lambda}_b)$ with respect to the components ϕ_u of the boson field, for $u \in \mathbf{\Lambda}_b$. The following definition extends this calculus by providing the standard Grassmann analogue of differentiation with respect to the fermion field (see, e.g., [3, 13, 16]).

Definition 2.3. For $u \in \mathbf{\Lambda}_f$, the linear map $i_u : \mathcal{N}(\mathbf{\Lambda}) \rightarrow \mathcal{N}(\mathbf{\Lambda})$ is defined uniquely by the conditions:

- (a) $i_u(f\psi^y) = fi_u\psi^y$ for $f \in \mathcal{R}(\mathbf{\Lambda}_b)$, $y \in \mathbf{\Lambda}_f^*$,
- (b) i_u acts as an anti-derivation on products of factors of ψ , namely $i_u(\psi^{y_1}\psi^{y_2}) = (i_u\psi^{y_1})\psi^{y_2} + (-1)^{p_1}\psi^{y_1}(i_u\psi^{y_2})$, for $y_1, y_2 \in \mathbf{\Lambda}_f^*$ and p_1 the length of y_1 , and

(c) $i_u \psi_v = \delta_{u,v}$ for $u, v \in \Lambda$, where the right-hand side is the Kronecker delta. It is natural, and also standard, to write

$$i_u = \frac{\partial}{\partial \psi_u}. \quad (2.12)$$

By (b) and (c), these operators anti-commute with each other: $i_u i_v = -i_v i_u$.

Suppose that there is a bijection $x \mapsto x'$ between a subset of Λ and Λ' . The elements of Λ where the bijection is not defined are called *external*; they do not participate in any integrations. We extend the matrices $\mathbf{C}_b, \mathbf{C}_f$ to $\Lambda_b \times \Lambda_b$ and $\Lambda_f \times \Lambda_f$, respectively, by setting $\mathbf{C}_{b,u',v'} = \mathbf{C}_{f,u',v'} = 0$ when u' or v' is undefined. We write \mathbf{C} for the pair $\mathbf{C}_b, \mathbf{C}_f$. The Laplacian operator $\Delta_{\mathbf{C}} : \mathcal{N}(\Lambda) \rightarrow \mathcal{N}(\Lambda)$ is then defined by

$$\Delta_{\mathbf{C}} = \sum_{u,v \in \Lambda_b} \mathbf{C}_{b;u,v} \frac{\partial}{\partial \phi_u} \frac{\partial}{\partial \phi_v} + \sum_{u,v \in \Lambda_f} \mathbf{C}_{f;u,v} \frac{\partial}{\partial \psi_u} \frac{\partial}{\partial \psi_v}, \quad (2.13)$$

where the first term on the right-hand side acts only on the coefficients $F_y(\phi)$ of $F \in \mathcal{N}$, while the second acts only on the fermionic part ψ^y .

2.6 Gaussian integration and the convolution property

Example 2.4. For a bounded function f defined on \mathbb{R} and a probability measure μ on \mathbb{R} , we can define the convolution $\mu \star f(x) = \int f(x+y) d\mu(y)$. The map $f \mapsto \mu \star f$ is the composition of the map $(\theta f)(x, y) = f(x+y)$ followed by integrating y with respect to μ . The map θ maps a function of one variable to a function of two variables.

The following definition implements the construction of Example 2.4 in the context of the algebra \mathcal{N} . To avoid simultaneously using ϕ to denote a function on Λ_b and a function on the larger space $\Lambda_b \sqcup \Lambda'_b$, we replace $\phi : \Lambda_b \sqcup \Lambda'_b \rightarrow \mathbb{C}$ by the notation

$$\phi \sqcup \xi : \Lambda_b \sqcup \Lambda'_b \rightarrow \mathbb{C}, \quad (2.14)$$

where $(\phi \sqcup \xi)_x = \phi_x$ and $(\phi \sqcup \xi)_{x'} = \xi_{x'}$. The algebra $\mathcal{N}(\Lambda)$ is a subset of $\mathcal{N}(\Lambda \sqcup \Lambda')$.

Definition 2.5. Given $t \in \mathbb{C}$, we define the algebra homomorphism $\theta_t : \mathcal{N}(\Lambda) \rightarrow \mathcal{N}(\Lambda \sqcup \Lambda')$ to be the unique algebra homomorphism which obeys:

- (a) the action on generators $\theta_t \psi_y = \psi_y + t \psi_{y'}$, for $y \in \Lambda_f$, and
- (b) the action on coefficients $(\theta_t f)(\phi) = f(\phi + t\xi)$, for $f \in \mathcal{R}(\Lambda_b)$.

If x' or y' is not defined, as discussed above (2.13), then the associated $\xi_{x'}$, $\psi_{y'}$ is set equal to zero. We write $\theta = \theta_1$.

The following proposition states a convolution property of Gaussian integrals that is at the heart of the renormalisation group method.

Proposition 2.6. For covariances $\mathbf{C}_1, \mathbf{C}_2$ and for $F \in \mathcal{N}(\Lambda)$ such that both sides of (2.15) are well-defined,

$$(\mathbb{E}_{\mathbf{C}_2} \theta \circ \mathbb{E}_{\mathbf{C}_1} \theta) F = \mathbb{E}_{\mathbf{C}_2 + \mathbf{C}_1} \theta F. \quad (2.15)$$

Moreover, if $P \in \mathcal{N}(\Lambda)$ is a polynomial of finite degree, as in Definition 2.1, then

$$\mathbb{E}_{\mathbf{C}} \theta P = e^{\frac{1}{2} \Delta_{\mathbf{C}}} P. \quad (2.16)$$

The identity (2.15) follows immediately from (2.16) for polynomial F , but (2.15) holds more generally. A proof of Proposition 2.6 is given in Section 4. The convolution property (2.15) is standard (see, e.g., [13] for the purely fermionic version), but our proof follows the approach in [6] which extends the familiar connection (2.16) between Gaussian integration and the Laplacian to the mixed bosonic-fermionic integral.

The formulas (2.16) and (2.13) compute moments. For example, if we take $P = \phi_u \phi_v$ and after evaluation of (2.16) set $\phi = 0$, the result is $\mathbb{E}_{\mathbf{C}} \xi_u \xi_v = \mathbf{C}_{b;u,v}$. Similarly, by taking $P = \psi_u \psi_v$, we obtain $\mathbb{E}_{\mathbf{C}} \psi_u \psi_v = -\mathbf{C}_{f;u,v}$. Thus (2.16) is a generalisation of Wick's theorem (see, e.g., [7, Lemma 2.3]), which is the standard formula for moments of a Gaussian measure.

2.7 Conjugate fermion field

Suppose that \mathbf{A}'_f has even cardinality $2M_f$, so the Grassmann generators can be written in a list as $\bar{\psi}_1, \psi_1, \dots, \bar{\psi}_{M_f}, \psi_{M_f}$, or, more compactly, as $(\bar{\psi}_k, \psi_k)_{k=1, \dots, M_f}$. For the Grassmann generators, there is not a notion of complex conjugation, so here the bars are used only as a notational device to list the generators in pairs. However, we will still refer to the pairs of generators as conjugate generators (and see Section 2.9 below). We use the order $\bar{\psi}_1, \psi_1, \bar{\psi}_2, \psi_2, \dots, \bar{\psi}_{M_f}, \psi_{M_f}$ for the generators in the definition of Grassmann integration in Definition 2.2.

Let A_f be an invertible symmetric $M_f \times M_f$ matrix, with $A_f^{-1} = C_f$. We define the matrix \mathbf{A}_f and its inverse matrix \mathbf{C}_f by

$$\mathbf{A}_f = \begin{pmatrix} 0 & A_f \\ -A_f^T & 0 \end{pmatrix}, \quad \mathbf{C}_f = \begin{pmatrix} 0 & -C_f^T \\ C_f & 0 \end{pmatrix}, \quad (2.17)$$

with the rows and columns labelled by $\psi_1, \dots, \psi_{M_f}, \bar{\psi}_1, \dots, \bar{\psi}_{M_f}$. Then S_f of (2.3) becomes

$$S_f = \sum_{k,l=1}^{M_f} A_{f;k,l} \psi_k \bar{\psi}_l \quad (2.18)$$

and the normalisation constant N_f of (2.5) is

$$N_f = (\det \mathbf{C}_f)^{1/2} = \det C_f. \quad (2.19)$$

For F a monomial in the Grassmann generators, let $J_F = \mathbb{E}_{\mathbf{C}_f} F$. The evaluation of the Grassmann integral J_F is standard (see, e.g., [16, Lemma B.7] or [7, Proposition 4.1]). In particular, $J_F = 1$ when $F = 1$, $J_F = 0$ when $F = \prod_{r=1}^p \bar{\psi}_{i_r} \prod_{s=1}^q \psi_{j_s}$ with $p \neq q$, and

$$J_F = \det C_{f;k_1, \dots, k_p; l_1, \dots, l_p}, \quad (2.20)$$

when $F = \bar{\psi}_{k_1} \psi_{l_1} \cdots \bar{\psi}_{k_p} \psi_{l_p}$, where $C_{f;k_1, \dots, k_p; l_1, \dots, l_p}$ is the $p \times p$ matrix whose r, s element is $C_{f;k_r, l_s}$. In particular,

$$\mathbb{E}_{\mathbf{C}_f} \bar{\psi}_k \psi_l = C_{f;k,l}, \quad (2.21)$$

and C_f is the covariance of the conjugate fermion field.

Conjugate fermion fields will be needed in Proposition 3.19 below.

2.8 Complex boson field

We now discuss a way to accommodate complex boson fields within the formalism. The boson field ϕ may include several species of fields, including real external fields which behave as constants during integration, and a complex field which does get integrated. To describe the latter, we suppose that Λ'_b has even cardinality $2M_b$ and write the field as $u_1, v_1, \dots, u_{M_b}, v_{M_b}$. Then, for $k = 1, \dots, M_b$, we define

$$\phi_k = u_k + iv_k, \quad \bar{\phi}_k = u_k - iv_k. \quad (2.22)$$

The boson field then corresponds to a complex field $(\bar{\phi}_k, \phi_k)_{k=1, \dots, M_b}$. Define

$$\frac{\partial}{\partial \phi_k} = \frac{1}{2} \left(\frac{\partial}{\partial u_k} - i \frac{\partial}{\partial v_k} \right), \quad \frac{\partial}{\partial \bar{\phi}_k} = \frac{1}{2} \left(\frac{\partial}{\partial u_k} + i \frac{\partial}{\partial v_k} \right). \quad (2.23)$$

By definition, these obey, for $k, l = 1, \dots, M_b$,

$$\frac{\partial \phi_k}{\partial \phi_l} = \frac{\partial \bar{\phi}_k}{\partial \bar{\phi}_l} = \delta_{k,l}, \quad \frac{\partial \phi_k}{\partial \bar{\phi}_l} = \frac{\partial \bar{\phi}_k}{\partial \phi_l} = 0. \quad (2.24)$$

Let A_b be a real invertible symmetric $M_b \times M_b$ matrix, with $A_b^{-1} = C_b$. We define the matrix \mathbf{A}_b and its inverse matrix \mathbf{C}_b by

$$\mathbf{A}_b = 2 \begin{pmatrix} A_b & 0 \\ 0 & A_b \end{pmatrix}, \quad \mathbf{C}_b = \frac{1}{2} \begin{pmatrix} C_b & 0 \\ 0 & C_b \end{pmatrix}, \quad (2.25)$$

with the rows and columns labelled by the real and imaginary parts $u_1, \dots, u_{M_b}, v_1, \dots, v_{M_b}$ of the complex boson field. Then S_b of (2.6) becomes

$$S_b = \sum_{k,l=1}^{M_b} A_{b;k,l} \phi_k \bar{\phi}_l \quad (2.26)$$

and the normalisation constant N_b of (2.8) is

$$N_b = (\det(2\pi \mathbf{C}_b))^{-1/2} = (\det(\pi C_b))^{-1}. \quad (2.27)$$

For $K \in \mathcal{N}(\Lambda \sqcup \Lambda'_b)$, the Gaussian integral $I_K = \mathbb{E}_{\mathbf{C}_b} K$ can equivalently be written as the complex Gaussian integral

$$I_K = \int_{\mathbb{C}^{M_b}} K d\mu_{C_b} \quad \text{with} \quad d\mu_{C_b} = N'_b e^{-S_b} \prod_{k=1}^{M_b} \frac{d\bar{\phi}_k d\phi_k}{2\pi i}, \quad (2.28)$$

where $d\bar{\phi}_k d\phi_k$ is by definition equal to $2idu_k dv_k$, where K is considered as a function of ϕ instead of as a function of the real and imaginary parts, and where the normalisation constant is

$$(N'_b)^{-1} = \int_{\mathbb{C}^{M_b}} e^{-S_b} \prod_{k=1}^{M_b} \frac{d\bar{\phi}_k d\phi_k}{2\pi i} = \frac{1}{\det C_b}. \quad (2.29)$$

The factors of 2 in (2.25) are included so that

$$\mathbb{E}_{\mathbf{C}_b} \bar{\phi}_k \phi_l = C_{b;k,l}, \quad (2.30)$$

and thus we call C_b the covariance of the complex boson field. Expectations of $\phi\phi$ and $\bar{\phi}\bar{\phi}$ are zero. More generally, expectations of products of factors of ϕ and $\bar{\phi}$ can be evaluated using (2.16) together with

$$\frac{1}{2}\Delta_C = \sum_{k,l=1}^{M_b} C_{b;k,l} \frac{\partial}{\partial \phi_k} \frac{\partial}{\partial \bar{\phi}_l} + \sum_{k,l=1}^{M_f} C_{f;k,l} \frac{\partial}{\partial \psi_k} \frac{\partial}{\partial \bar{\psi}_l}, \quad (2.31)$$

where we computed the Laplacian (2.13) using (2.23) and (2.25).

2.9 Differential forms

Suppose we are in the setting of the conjugate fermion field and complex boson field of Sections 2.7–2.8, and that $M_f = M_b = M$. Let

$$S_A = S_b + S_f. \quad (2.32)$$

Now (2.11) can be written as

$$\mathbb{E}_C KF = I_K J_F = N'_b N_f \int e^{-S_A} KF, \quad (2.33)$$

where the Lebesgue measure $D\phi$ has been omitted intentionally from the right-hand side. The reason for this omission makes use of a specific choice of Grassmann generators, as follows.

We choose as Grassmann generators the 1-forms

$$\begin{aligned} \psi_k &= \frac{1}{(2\pi i)^{1/2}} d\phi_k = \frac{1}{(2\pi i)^{1/2}} (du_k + idv_k), \\ \bar{\psi}_k &= \frac{1}{(2\pi i)^{1/2}} d\bar{\phi}_k = \frac{1}{(2\pi i)^{1/2}} (du_k - idv_k), \end{aligned} \quad (2.34)$$

where we fix a choice of square root of $2\pi i$ once and for all. Multiplication of generators is via the standard anti-commuting wedge product for differential forms (see, e.g., [15]); the wedges are left implicit in what follows. The 1-forms generate the Grassmann algebra of differential forms. In this case the complex conjugation that acts on the boson field at the same time interchanges ψ_k and $\bar{\psi}_k$, but there are no relations other than anti-commutativity linking the generators of the Grassmann algebra. Now (2.32) becomes the differential form

$$S_A = \sum_{k,l=1}^M \left(A_{b;k,l} \phi_k \bar{\phi}_l + \frac{1}{2\pi i} A_{f;k,l} d\phi_k d\bar{\phi}_l \right). \quad (2.35)$$

The theory of Gaussian integration in this setting is developed in [7]. In particular, it follows from [7, Proposition 4.1] that when we interpret the fermionic part of e^{-S_f} as the differential form $\sum_{n=0}^{\infty} \frac{1}{n!} (-S_f)^n$ (the series truncates due to anti-commutativity), then standard integration of differential forms gives again

$$\mathbb{E}_C KF = I_K J_F. \quad (2.36)$$

Thus Grassmann integral and the standard integration of differential forms become the same thing. In the formalism of differential forms, the omitted Lebesgue measure is supplied by the volume

form $\prod_{k=1}^M d\bar{\phi}_k d\phi_k$ arising from the expansion of e^{-S_f} . Earlier, we defined $d\bar{\phi}_k d\phi_k$ to be $2idu_k dv_k$ because by (2.22) this is the wedge product $d\bar{\phi}_k d\phi_k$.

The above shows that the algebra of differential forms and the form integration used in [7] is a special case of the construction of Sections 2.7–2.8. We do not need this special case in this paper, but it plays an important role in [1, 2].

2.10 Supersymmetry

The field theories discussed in [7] and [2] have an additional property of *supersymmetry*: a symmetry between bosons and fermions. A discussion of supersymmetry can be found in [7, Section 6]. The field theory becomes supersymmetric by choosing the boson and fermion covariances to be equal: $C_b = C_f = C$. Then

$$N'_b N_f = \frac{\det C_f}{\det C_b} = 1, \quad (2.37)$$

and, with $A = C^{-1}$, (2.32) becomes

$$S_A = \sum_{u,v \in \Lambda} A_{u,v} (\phi_u \bar{\phi}_v + \psi_u \bar{\psi}_v). \quad (2.38)$$

Also, in view of (2.37), the normalisation constants cancel in (2.11), which becomes

$$\mathbb{E}_C K F = \left(\int_{\mathbb{C}^M} K e^{-S_b} \prod_{k=1}^{M_b} d\bar{\phi}_k d\phi_k \right) \left(\int_{\Lambda_f} e^{-S_f} F \right) = I_K J_F. \quad (2.39)$$

The Laplacian (2.31) now simplifies to

$$\Delta_C = \sum_{k,l=1}^{M_b} C_{k,l} \left(\frac{\partial}{\partial \phi_k} \frac{\partial}{\partial \bar{\phi}_l} + \frac{\partial}{\partial \psi_k} \frac{\partial}{\partial \bar{\psi}_l} \right), \quad (2.40)$$

and from (2.39) we obtain

$$\mathbb{E}_C \bar{\phi}_k \phi_l = \mathbb{E}_C \bar{\psi}_k \psi_l = C_{kl}. \quad (2.41)$$

2.11 Factorisation property of the expectation

We now present a factorisation property of the expectation that is needed in [9]. We formulate the factorisation property in the supersymmetric setting of Section 2.10 for simplicity, although it does hold more generally.

Let $\Lambda = \Lambda_b = \Lambda_f$, and let $X \subset \Lambda$. We define $\mathcal{N}(X)$ to be the set of all $F = \sum_{y \in \Lambda^*} F_y \psi^y \in \mathcal{N}$ such that $F_y = 0$ if any component of y is not in X , and such that, for all y , F_y does not depend on ϕ_x for any $x \notin X$. Similarly, given $X' \subset \Lambda'$, we define $\mathcal{N}(\Lambda \sqcup X')$ as those F that only depend on the fermion and boson fields indexed by $\Lambda \sqcup X'$.

Proposition 2.7. *Let $X, Y \subset \Lambda$, let $F_1(X) \in \mathcal{N}(\Lambda \sqcup X')$, $F_2(Y) \in \mathcal{N}(\Lambda \sqcup Y')$, and suppose that $C_{x',y'} = 0$ whenever $x' \in X'$, $y' \in Y'$. Then the expectation \mathbb{E}_C has the factorisation property:*

$$\mathbb{E}_C (F_1(X) F_2(Y)) = (\mathbb{E}_C F_1(X)) (\mathbb{E}_C F_2(Y)). \quad (2.42)$$

Proof. By linearity of the expectation, it suffices to consider the case where $F_1(X)$ is of the form $f_1\psi^x$ where f_1 depends only on the boson field in $\Lambda \sqcup X'$ and $x \in (X')^*$, and where $F_2(Y)$ is of the form $f_2\psi^y$ where f_2 depends only on the boson field in $\Lambda \sqcup Y'_b$ and $y \in (Y')^*$. According to (2.39), the expectation factors as

$$\mathbb{E}_C f_1 \psi^x f_2 \psi^y = (\mathbb{E}_C f_1 f_2)(\mathbb{E}_C \psi^x \psi^y), \quad (2.43)$$

where the first expectation on the right-hand side is a bosonic expectation with covariance matrix C , while the second is a fermionic expectation which is equal to a determinant of a submatrix of C taken from rows and columns labelled by the points in x and y .

By assumption, the covariance matrix elements vanish for rows and columns labelled by points in X and Y , respectively. It is a standard fact that uncorrelated Gaussian random vectors are independent [12], and hence $\mathbb{E}_C f_1 f_2 = (\mathbb{E}_C f_1)(\mathbb{E}_C f_2)$. Also by assumption, the determinant yielding the fermion expectation is the determinant of a block diagonal matrix, so also factors to give $(\mathbb{E}_C \psi^x \psi^y) = (\mathbb{E}_C \psi^x)(\mathbb{E}_C \psi^y)$. This completes the proof. ■

3 The T_ϕ semi-norm

3.1 Motivation

In the progressive integrations carried out in the renormalisation group approach, it is necessary to estimate how the size of the result of an integration compares with the size of the integrand. When integrating real-valued functions of real variables, the inequality

$$\left| \int f(x) dx \right| \leq \int |f(x)| dx \quad (3.1)$$

is fundamental. We need an analogue of (3.1) for the Gaussian integral $\mathbb{E}_C : \mathcal{N}(\Lambda \sqcup \Lambda') \rightarrow \mathcal{N}(\Lambda)$. In particular, we need to define norms (or semi-norms) so that $\mathcal{N}(\Lambda \sqcup \Lambda')$ and $\mathcal{N}(\Lambda)$ become normed algebras. The norms we define here emerge from a long history going back to [10]; other norms in the purely fermionic context are developed in [14].

We will choose Λ_b and Λ_f each to consist of disjoint unions of copies of the discrete d -dimensional torus of side length R^m , namely

$$\Lambda = \mathbb{Z}^d / (R^m \mathbb{Z}^d), \quad (3.2)$$

where $R \geq 2$ and $m \geq 1$ are integers. The copies allow different kinds of fields to be united in ϕ, ψ . For example, suppose there are three species of field: the first species is a real external boson field a , the second species is a complex boson field as in Section 2.8, and the third species is a conjugate fermion field as in Section 2.7. We choose $\Lambda_b = \Lambda_1 \sqcup \Lambda_2 \sqcup \bar{\Lambda}_2$ and $\Lambda_f = \Lambda_3 \sqcup \bar{\Lambda}_3$, where each Λ_i and $\bar{\Lambda}_i$ is a copy of Λ . The fermion field $(\psi_v)_{v \in \Lambda}$ is $\psi_{y \in \Lambda_f}$ restricted to $y \in \Lambda_3$, the fermion field $(\bar{\psi}_v)_{v \in \Lambda}$ is $\psi_{y \in \Lambda_f}$ restricted to $y \in \bar{\Lambda}_3$, the complex boson field $(\bar{\phi}_v, \phi_v)_{v \in \Lambda}$ is the restriction of ϕ to $\Lambda_2 \sqcup \bar{\Lambda}_2$, and the external boson field $(a_v)_{v \in \Lambda}$ is $\phi_{x \in \Lambda_b}$ restricted to $x \in \Lambda_1$. For $x \in \Lambda_2$ let \bar{x} be the corresponding point in the copy $\bar{\Lambda}_2$. The restriction of ϕ to $\Lambda_2 \sqcup \bar{\Lambda}_2$ is a complex field $\phi = u + iv$ as defined in Section 2.8 if and only if

$$\phi_{\bar{x}} = \bar{\phi}_x, \quad x \in \Lambda_2. \quad (3.3)$$

Given $u \in \Lambda$, an example of an element of $\mathcal{N}(\Lambda)$ is

$$K = e^{-a_u(\phi_u \bar{\phi}_u + \psi_u \bar{\psi}_u)}; \quad (3.4)$$

here the appearance of the boson field on the right-hand side is a notational convenience analogous to referring to the function e^{-t} (rather than $e^{-(\cdot)}$). Functions of the fermion field are to be understood as elements of \mathcal{N} via Taylor expansion in powers of the fermion field. Due to anti-commutativity and the finite index set for the fermion field, such Taylor series always truncate to polynomials in the fermion field. For (3.4), the Taylor polynomial is

$$K = e^{-a_u(\phi_u \bar{\phi}_u + \psi_u \bar{\psi}_u)} = e^{-a_u \phi_u \bar{\phi}_u} (1 - a_u \psi_u \bar{\psi}_u). \quad (3.5)$$

For functions of products of even numbers of ψ factors, which are the only kind we will encounter, there is no sign ambiguity in the Taylor expansion. We replace ϕ by $\phi + \xi$ and a by $a + \hat{a}$ and expand in powers of $\hat{a}_u, \xi_u, \bar{\xi}_u$. We use the set $\Lambda \sqcup \bar{\Lambda}$ to keep track of factors ξ_u versus $\bar{\xi}_u$, by writing, e.g., $\xi^x = \xi_u \xi_u \bar{\xi}_u \bar{\xi}_u$ for $x = (u, u, \bar{u}, u, \bar{u}) \in (\Lambda \sqcup \bar{\Lambda})^*$, and similarly for the fermion field. This leads to the expansion, for ϕ replaced by $\phi + \xi$ and a by $a + \hat{a}$,

$$K = \sum_{w \circ x \circ y \in \bar{\Lambda}^*} \frac{1}{w!x!y!} K_{w,x,y}(a_u, \bar{\phi}_u, \phi_u) \hat{a}^w \xi^x \psi^y, \quad (3.6)$$

where $\bar{\Lambda}^*$ denotes the set of sequences obtained by concatenating sequences $w \in \Lambda^*$, $x \in (\Lambda \sqcup \bar{\Lambda})^*$, and $y \in (\Lambda \sqcup \bar{\Lambda})^*$, and where the coefficients $K_{w,x,y}$ are symmetric in the elements of w and the elements of x , and anti-symmetric in the elements of y .

The semi-norm of K , for fixed $a_u, \bar{\phi}_u, \phi_u$, is defined in terms of the coefficients $K_{w,x,y}(a_u, \bar{\phi}_u, \phi_u)$. These coefficients eventually vanish once the sequence y has length exceeding twice the cardinality of Λ . In general, the coefficients will be non-zero for infinitely many values of w, x , but the semi-norm will examine only those with length of w and length of x at most p_N for a fixed choice of the parameter p_N . The semi-norm is designed to be used in conjunction with integration, in which the fields ξ and ψ are integrated with respect to a Gaussian measure with covariance C . According to (2.21) and (2.30), these fields may be considered to have typical size equal to the square root of the covariance $C_{u,u}$. In our applications, the external field will also have a natural size A_u . This motivates us to define the semi-norm of K to be the result of replacing $\hat{a}^w \xi^x \psi^y$, in each term in the sum in (3.6), by a test function $g_{w,x,y}$ whose size is of order $A^w C_{u,u}^{(x+y)/2}$, where the superscripts are to be interpreted as the sequence lengths. This semi-norm should then measure the size of K when evaluated on typical fields. In addition, we want the semi-norm to capture the size of typical field gradients. From (2.30), we conclude that $\mathbb{E}_{C_b}(|\nabla \phi_u|^2) = \nabla^2 C_{u,u}$, suggesting a restriction to test functions whose gradients reflect this behaviour.

The precise definition of the semi-norm, given below, is more general than the above in several respects. It allows the possibility of more “species” of field than the three boson, fermion and external fields above, and allows scalar, complex, and multi-component fields. It allows distinction between the size of the test functions in its components corresponding to different field species, and leaves flexible the choice of weights governing the test functions since in our applications typical field size will sometimes be governed by an interaction rather than the covariance of a Gaussian integral.

In the remainder of Section 3, we define the T_ϕ semi-norm on \mathcal{N} and state and develop its properties. Most proofs are deferred to Sections 5–6.

3.2 Sequence spaces

The sets Λ_b and Λ_f are required to have the following particular structure. First, Λ_b decomposes into a disjoint union of sets $\Lambda_b^{(i)}$, for $i = 1, \dots, s_b$, corresponding to s_b distinct boson field *species*. Each sets $\Lambda_b^{(i)}$ is either $\Lambda \sqcup \bar{\Lambda}$ (for a species of complex field) or is the disjoint union of c_b^i copies of Λ (for a field species with c_b^i real components). The set Λ_f has the same structure, but with a possibly different number s_f of species which can also have components. Then, as before, we set $\Lambda = \Lambda_b \sqcup \Lambda_f$, and Λ^* is the corresponding set of sequences. Each $u \in \Lambda$ thus carries a species label $i = i(u) \in \mathbf{s} = \{1, \dots, s\}$, where $s = s_b + s_f$.

Of specific interest is the subset $\vec{\Lambda}^*$ of Λ^* , which consists of sequences whose species labels are ordered in such a way that the first elements of $z \in \vec{\Lambda}^*$ are of species $\Lambda_b^{(1)}$, the next are of species $\Lambda_b^{(2)}$, and so on until the boson species have been exhausted, and then subsequent elements are first of species $\Lambda_f^{(1)}$, then $\Lambda_f^{(2)}$, and so on. We also define $\vec{\Lambda}_b^*$ and $\vec{\Lambda}_f^*$ to be the subsets of $\vec{\Lambda}^*$ consisting of only boson or only fermion species. There is a canonical bijection between $\vec{\Lambda}^*$ and the Cartesian product $\Lambda_b^{(1)} \times \dots \times \Lambda_b^{(s_b)} \times \Lambda_f^{(1)} \times \dots \times \Lambda_f^{(s_f)}$, given by the correspondence in which a single sequence in $\vec{\Lambda}^*$ is regarded as a collection of subsequences of each species. We will sometimes blur the distinction between $\vec{\Lambda}^*$ and the Cartesian product in what follows. In $\vec{\Lambda}^*$, concatenation $z' \circ z''$ of two sequences is defined by concatenation of each of the individual species subsequences. For $z \in \vec{\Lambda}^*$ we write $z! = \prod_{i=1}^s z_i!$ where the product is over species and $z_i!$ denotes the factorial of the length of the species- i subsequence of z . For $r \geq 0$, we write $\vec{\Lambda}^{(r)}$ for the subset of $\vec{\Lambda}^*$ consisting of sequences of length r , with the degenerate case $\vec{\Lambda}^{(0)} = \{\emptyset\}$.

3.3 Test functions

A *test function* is a function $g : \vec{\Lambda}^* \rightarrow \mathbb{C}$. In particular, even when there are complex fields, no relation such as (3.3) is imposed on test functions. We will define a norm on the set of test functions as a weighted finite-difference version of a C^k norm.

First we need notation for multiple finite-difference derivatives. We write \mathcal{U} for the set $\{\pm e_1, \dots, \pm e_d\}$ of $2d$ positive and negative unit vectors in \mathbb{Z}^d . For a unit lattice vector $e \in \mathcal{U}$ and a function f on Λ the difference operator is given by $\nabla^e f_x = f_{x+e} - f_x$. When e is the negative of a standard unit vector ∇^e is the negative of a conventional backward derivative. To specify a higher order derivative we specify how many times each directional derivative occurs: let $A = \mathbb{N}_0^{\mathcal{U}}$ and for $\alpha \in A$ define

$$\nabla^\alpha = \prod_{e \in \mathcal{U}} \nabla^{\alpha_e}, \quad \nabla^0 = \text{Id}. \quad (3.7)$$

Since test functions are defined on $\vec{\Lambda}^*$ we have to go up one more level of sequences. Let $A = \mathcal{U}^*$. Derivatives are given by the finite-difference derivative ∇^α , where $\alpha \in A^*$; the component α_k of α specifies the finite-difference derivatives to be applied to the component z_k of $z \in \vec{\Lambda}^*$. For $r > 0$, we define $\mathcal{A}^{(r)} = A^r \times \vec{\Lambda}^{(r)}$, and in the degenerate case set $\mathcal{A}^{(0)} = \{\emptyset\}$. Let $\mathcal{A} = \cup_{r \geq 0} \mathcal{A}^{(r)}$. Thus an element of \mathcal{A} specifies a sequence z and a number of finite-difference derivatives to be applied component-wise in z .

Definition 3.1. Fix $p_{\mathcal{N}} \in \mathbb{N}_0 \cup \{+\infty\}$, and consider the set of test functions such that $g_z = 0$ whenever z has more than $p_{\mathcal{N}}$ boson components. Let $w : A \times \Lambda \rightarrow [0, \infty]$ be a function such that $w_{\alpha,z}$ depends on $z \in \Lambda$ only through its species $i(z)$. For $r > 0$ and $(\alpha, z) \in \mathcal{A}^{(r)}$ we write $w_{\alpha,z} = \prod_{j=1}^r w_{\alpha_j, z_j}$, and we set $w_{\varnothing} = 1$ in the degenerate case $r = 0$. We define the Φ norm on test functions by

$$\|g\|_{\Phi} = \sup_{(\alpha, z) \in \mathcal{A}} w_{\alpha, z}^{-1} |\nabla^{\alpha} g_z|. \quad (3.8)$$

Let $g^{(r)} : \vec{\Lambda}^{(r)} \rightarrow \mathbb{C}$ denote the restriction of $g : \vec{\Lambda}^* \rightarrow \mathbb{C}$ to $\vec{\Lambda}^{(r)}$. The Φ norm induces the $\Phi^{(r)}$ norm on these restricted test functions by

$$\|g^{(r)}\|_{\Phi^{(r)}} = \sup_{(\alpha, z) \in \mathcal{A}^{(r)}} w_{\alpha, z}^{-1} |\nabla^{\alpha} g_z^{(r)}|, \quad (3.9)$$

with $0^{-1} = \infty$ and $\infty^{-1} = 0$, and with

$$\|g\|_{\Phi} = \sup_{r \geq 0} \|g^{(r)}\|_{\Phi^{(r)}}. \quad (3.10)$$

When it is important to make the dependence on w explicit we write $\Phi(w)$ and $\Phi^{(r)}(w)$.

As an instance of restriction, suppose that there is just one species of field, namely a single complex boson field $(\bar{\phi}_x, \phi_x)_{x \in \Lambda}$. We may regard this field as a test function by extending it to be the zero function on sequences in $\vec{\Lambda}^*$ of length different from 1. This special case will frequently be relevant for us.

Example 3.2. Fix $p_{\Phi} \geq 0$ and for each species i fix $\mathfrak{h}_i > 0$. Let R be the constant of (3.2). The choice of weights

$$w_{\alpha_j, z_j}^{-1} = \begin{cases} \mathfrak{h}_{i(z_j)}^{-z_j} R^{\alpha_j} & \text{if } |\alpha|_1 \leq p_{\Phi} \\ 0 & \text{if } |\alpha|_1 > p_{\Phi} \end{cases} \quad (3.11)$$

defines the normed space $\Phi(\mathfrak{h})$. The z_j in the exponent on the right-hand side represents the length of the sequence z_j , and the α_j in the power of R represents the order $|\alpha_j|_1$ of the derivative ∇^{α_j} . Then test functions in the unit ball $B(\Phi)$ of Φ are those which obey the estimate

$$|\nabla^{\alpha} g_z| \leq \mathfrak{h}^z R^{-\alpha} \quad (3.12)$$

for all z with at most $p_{\mathcal{N}}$ boson components and for all $|\alpha|_1 \leq p_{\Phi}$. Here \mathfrak{h}^{-z} is an abbreviation for $\prod_j \mathfrak{h}_{i(z_j)}^{-z_j}$. The estimate (3.12) means that g is approximately constant on regions whose diameter is small compared to R .

Let $\vec{r} \in \mathbb{N}_0^s$ and let $\Phi^{(\vec{r})}$ denote the restriction of Φ to test functions defined on the subset of $\vec{\Lambda}^*$ consisting of sequences with exactly r_i components of species i for each $i = 1, \dots, s$. Given \vec{r} , $g' \in \Phi^{(\vec{r})}$, and $g'' \in \Phi$, we define $g \in \Phi$ by setting $g_z = g'_{z'} g''_{z''}$ for $z = z' \circ z''$, with $g_z = 0$ whenever z has fewer than r_i elements of species i for any i . It follows from the definition of the norm that

$$\|g\|_{\Phi} \leq \|g'\|_{\Phi^{(\vec{r})}} \|g''\|_{\Phi}, \quad (3.13)$$

and we will use this fact later. Here it is the fact that $g' \in \Phi^{(\vec{r})}$ which provides a unique decomposition $z = z' \circ z''$ to make g well defined. A similar inequality is obtained whenever a unique decomposition is specified. For example, suppose that we designate some field species as prime species and some as double prime. Then z can be decomposed in a unique way as (z', z'') and if we define a test function g by $g_z = g'_z g''_{z''}$, then it follows from the definition of the norm that

$$\|g\|_\Phi \leq \|g'\|_\Phi \|g''\|_\Phi. \quad (3.14)$$

3.4 Definition of the T_ϕ semi-norm

Given $F \in \mathcal{N}(\Lambda)$, $x = (x_1, \dots, x_p) \in \Lambda_b$, $y \in \Lambda_f$, and a boson field $\phi \in \mathbb{R}^{\Lambda_b}$, we write

$$F_{x,y}(\phi) = \frac{\partial^p F_y(\phi)}{\partial \phi_{x_p} \cdots \partial \phi_{x_1}}. \quad (3.15)$$

We are writing the boson field as an element of \mathbb{R}^{Λ_b} for simplicity, but our intention is to include the possibility of complex species and for such species derivatives are with respect to ϕ_{x_i} or $\bar{\phi}_{x_i}$ depending on whether x_i is an element of Λ or $\bar{\Lambda}$. This point will be made more explicit in Section 3.5 below.

Definition 3.3. For a test function $g : \bar{\Lambda}^* \rightarrow \mathbb{C}$, for $F \in \mathcal{N}(\Lambda)$, and for $\phi \in \mathbb{R}^{\Lambda_b}$, we define the *pairing*

$$\langle F, g \rangle_\phi = \sum_{z \in \bar{\Lambda}^*} \frac{1}{z!} F_z(\phi) g_z = \sum_{x \in \bar{\Lambda}_b^*} \sum_{y \in \bar{\Lambda}_f^*} \frac{1}{x! y!} F_{x,y}(\phi) g_{x,y}, \quad (3.16)$$

and the T_ϕ semi-norm

$$\|F\|_{T_\phi} = \sup_{g \in B(\Phi)} |\langle F, g \rangle_\phi|, \quad (3.17)$$

where $B(\Phi)$ denotes the unit ball in the space Φ of test functions.

By definition, $F_{x,y}$ is symmetric under permutations within each subsequence of x having the same species, and is similarly antisymmetric in y . This symmetry is reflected by a corresponding property of the pairing. To develop this idea, we begin with the following definition.

Definition 3.4. For $z \in \bar{\Lambda}^{(r)}$, let $\vec{\Sigma}_z$ denote the set of permutations of $1, \dots, r$ that preserve the order of the species of z . For $\sigma \in \vec{\Sigma}_z$ we define $\sigma z \in \bar{\Lambda}^{(r)}$ by $(\sigma z)_i = z_{\sigma(i)}$, and we use this to define a map $S : \Phi \rightarrow \Phi$ by

$$(Sg)_z = \frac{1}{z!} \sum_{\sigma \in \vec{\Sigma}_z} \text{sgn}(\sigma_f) g_{\sigma z}, \quad (3.18)$$

where σ_f denotes the restriction of σ to the fermion components of z and $\text{sgn}(\sigma_f)$ denotes the sign of this permutation.

Proposition 3.5. For $F \in \mathcal{N}(\Lambda)$, $g \in \Phi$, and $\phi \in \mathbb{R}^{\Lambda_b}$,

$$\langle F, g \rangle_\phi = \langle F, Sg \rangle_\phi. \quad (3.19)$$

Proof. By the above-mentioned symmetry, $F_z(\phi) = \text{sgn}(\sigma_f)F_{\sigma(z)}(\phi)$ for all $\sigma \in \vec{\Sigma}_z$. This implies that $F_z(\phi) = \frac{1}{z!} \sum_{\sigma \in \vec{\Sigma}_z} \text{sgn}(\sigma_f)F_{\sigma(z)}(\phi)$, and hence

$$\langle F, g \rangle_\phi = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} F_z(\phi) g_z = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} \frac{1}{z!} \sum_{\sigma \in \vec{\Sigma}_z} \text{sgn}(\sigma_f) F_{\sigma(z)}(\phi) g_z. \quad (3.20)$$

The sum over z is graded by sums over sequences of fixed length and species choices, and for z fixed within this gradation the set $\vec{\Sigma}_z$ is independent of z . It therefore makes sense to replace the summand within the sum over σ by an equivalent expression with z replaced by $\sigma^{-1}z$, and this does not change the sum. This gives

$$\langle F, g \rangle_\phi = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} \frac{1}{z!} \sum_{\sigma \in \vec{\Sigma}_z} \text{sgn}(\sigma_f) F_z(\phi) g_{\sigma^{-1}z} = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} F_z(\phi) \frac{1}{z!} \sum_{\sigma \in \vec{\Sigma}_z} \text{sgn}(\sigma_f) g_{\sigma^{-1}z}. \quad (3.21)$$

Since $\text{sgn}(\sigma_f) = \text{sgn}(\sigma_f^{-1})$, and since summing over σ is the same as summing over σ^{-1} , this gives the desired result. \blacksquare

Example 3.6. As a simple example of the zero-field pairing, for fixed points $x_i \in \Lambda$ and for $p \leq p_N$, let $F(\phi) = \prod_{i=1}^p \nabla^{\alpha_i} \phi_{x_i}$. Direct computation shows that (3.16) leads to

$$\langle F, g \rangle_0 = \nabla_{x_1}^{\alpha_1} \cdots \nabla_{x_p}^{\alpha_p} (Sg)_{x_1, \dots, x_p}. \quad (3.22)$$

The right-hand side is in general not the same as the corresponding expression with S omitted. This shows that the pairing has a symmetrising effect.

By definition,

$$\|F\|_{T_\phi} = \sup_{r \geq 0} \sup_{g^{(r)} \in B(\Phi^{(r)})} |\langle F, g^{(r)} \rangle_\phi|. \quad (3.23)$$

Note that $\|F\|_{T_\phi}$ is always at least as large as $|F_\emptyset|$ because this is the contribution from the empty sequence part g_\emptyset of the test function, corresponding to $r = 0$. The T_ϕ semi-norm has several attractive and useful properties. The most fundamental of these is the product property stated in the following proposition. Its proof is given in Section 5.1 below.

Proposition 3.7. *For $F, G \in \mathcal{N}$, $\|FG\|_{T_\phi} \leq \|F\|_{T_\phi} \|G\|_{T_\phi}$.*

Another property is the following proposition, which is proved in Section 5.2 below. In its statement, e^{-F} is defined by Taylor expansion in the fermion field. In general, this can introduce sign ambiguities, but the semi-norm is insensitive to these by (3.23). However, in our application in (3.29) below, no sign ambiguity arises.

Proposition 3.8. *Let $F \in \mathcal{N}$ and let F_\emptyset be the purely bosonic part of F . Then*

$$\|e^{-F}\|_{T_\phi} \leq e^{-2\text{Re}F_\emptyset(\phi) + \|F\|_{T_\phi}}. \quad (3.24)$$

3.5 Example for the T_ϕ semi-norm

For the next proposition, we consider the case $\Lambda_b = (\Lambda \sqcup \bar{\Lambda})$ and $\Lambda_f = (\Lambda \sqcup \bar{\Lambda})$, corresponding to a complex boson field $(\bar{\phi}, \phi)$ and a conjugate fermion field $(\bar{\psi}, \psi)$. We use the test function space $\Phi(\mathfrak{h})$ of Example 3.2, with its associated space $T_\phi(\mathfrak{h})$, where \mathfrak{h} takes the same value for all fields. For a complex boson field ϕ and $x \in \Lambda$, we define $\tau_x \in \mathcal{N}$ by

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \bar{\psi}_x. \quad (3.25)$$

We may regard ϕ_x as an element of \mathcal{N} . By definition its T_ϕ semi-norm is $\|\phi_x\|_{T_\phi(\mathfrak{h})} = |\phi_x| + \mathfrak{h}$. We may also regard the boson field ϕ as the test function obtained by extending to the zero function on sequences in $\bar{\Lambda}^*$ which do not consist of a single component in Λ_b ; then its norm is $\|\phi\|_\Phi$. Since $\mathfrak{h}^{-1}|\phi_x| \leq \|\phi\|_{\Phi(\mathfrak{h})}$ by definition, we have

$$\|\phi_x\|_{T_\phi} \leq \mathfrak{h} (1 + \|\phi\|_{\Phi(\mathfrak{h})}). \quad (3.26)$$

Proposition 3.9. *The $T_\phi(\mathfrak{h})$ semi-norm of τ_x obeys the identity*

$$\|\tau_x\|_{T_\phi(\mathfrak{h})} = (|\phi_x| + \mathfrak{h})^2 + \mathfrak{h}^2 \quad (3.27)$$

and the inequality

$$\|\tau_x\|_{T_\phi(\mathfrak{h})} \leq 3\mathfrak{h}^2(1 + \|\phi\|_{\Phi(\mathfrak{h})}^2). \quad (3.28)$$

Suppose that $a \in \mathbb{C}$ obeys $|\operatorname{Im} a| \leq \frac{1}{2}\operatorname{Re} a$. Given any real number q_2 , there is a constant q_1 (with $q_1 = O(q_2^2)$ as $q_2 \rightarrow \infty$) such that

$$\|e^{-a\tau_x^2}\|_{T_\phi(\mathfrak{h})} \leq e^{(\operatorname{Re} a)\mathfrak{h}^4(q_1 - q_2|\phi_x/h|^2)}. \quad (3.29)$$

Proof. By definition, $\tau_x = \phi_x \bar{\phi}_x + \psi_x \bar{\psi}_x$. Also by definition, the semi-norm of a sum of terms of different fermionic degree is the sum of the semi-norms, and hence

$$\|\tau_x\|_{T_\phi} = \|\phi_x \bar{\phi}_x\|_{T_\phi} + \|\psi_x \bar{\psi}_x\|_{T_\phi}. \quad (3.30)$$

By definition of the semi-norm,

$$\|\psi_x \bar{\psi}_x\|_{T_\phi(\mathfrak{h})} = \mathfrak{h}^2 \quad (3.31)$$

and

$$\|\phi_x \bar{\phi}_x\|_{T_\phi(\mathfrak{h})} = |\phi_x|^2 + |\phi_x| \mathfrak{h} + \mathfrak{h} |\bar{\phi}_x| + \mathfrak{h}^2 = (|\phi_x| + \mathfrak{h})^2. \quad (3.32)$$

This proves (3.27). We write $t = |\phi_x|/\mathfrak{h}$ and $P(t) = (t+1)^2 + 1$. Then

$$\|\tau_x\|_{T_\phi(\mathfrak{h})} = \mathfrak{h}^2 P(t). \quad (3.33)$$

Since $t \leq \|\phi\|_{\Phi(\mathfrak{h})}$ and $P(t) \leq 3(1+t^2)$, this gives

$$\|\tau_x\|_{T_\phi(\mathfrak{h})} \leq \mathfrak{h}^2 P(t) = \mathfrak{h}^2 P(\|\phi\|_{\Phi(\mathfrak{h})}) \leq 3\mathfrak{h}^2(1 + \|\phi\|_{\Phi(\mathfrak{h})}^2), \quad (3.34)$$

which proves (3.28).

Let $\alpha = \operatorname{Re} a$. By (3.34), the product property, and the fact that $|a| \leq \frac{3}{2}\alpha$ by assumption,

$$\|a\tau_x^2\|_{T_{\phi(\mathfrak{h})}} \leq |a| \mathfrak{h}^4 P(t)^2 \leq \frac{3}{2}\alpha \mathfrak{h}^4 P(t)^2. \quad (3.35)$$

By Proposition 3.8 and (3.35),

$$\|e^{-a\tau_x^2}\|_{T_{(a,\phi)}} \leq e^{-2\alpha|\phi_x|^4} e^{\frac{3}{2}\alpha \mathfrak{h}^4 P(t)^2} \leq e^{\alpha \mathfrak{h}^4 [-2t^4 + \frac{3}{2}P(t)^2]}. \quad (3.36)$$

Since P has leading term t^2 , given any real number q_2 there is a constant $q_1 = O(q_2^2)$ such that $-2t^4 + \frac{3}{2}P(t)^2 \leq q_1 - q_2 t^2$. (In fact, a quartic bound also holds, but this quadratic bound will suffice for our needs.) This gives (3.29), and completes the proof. \blacksquare

On the right-hand side of (3.28), the appearance of the norm $\|\phi\|_{\Phi(\mathfrak{h})}$ could be considered alarming, as this involves a supremum over the entire lattice and typical fields will be uncontrollably large in some regions of space. In our applications this difficulty will be overcome as follows. First, we need some definitions. For $X \subset \Lambda$ and any test function space Φ , we define a new norm on Φ by

$$\|g\|_{\Phi(X)} = \inf\{\|g - f\|_{\Phi} : f_z = 0 \text{ if all components of } z \in \vec{\Lambda}^* \text{ are in } X\}. \quad (3.37)$$

As in Section 2.11, we define

$$\mathcal{N}(X) = \{F \in \mathcal{N} : F_z = 0 \text{ if any component of } z \in \vec{\Lambda}^* \text{ is not in } X\}. \quad (3.38)$$

Then $\mathcal{N}(X)$ is a subspace of \mathcal{N} , and $\mathcal{N} = \mathcal{N}(\Lambda)$. Suppose now that $F \in \mathcal{N}(X)$. Changing the value of ϕ_x for $x \notin X$ has no effect on the pairing of F with any test function g and hence has no effect on any T_ϕ semi-norm of F . Thus, returning to (3.28), by taking the infimum over all possible redefinitions of ϕ off $X = \{x\}$, we can replace (3.28) by

$$\|\tau_x\|_{T_\phi(\mathfrak{h})} \leq 3\mathfrak{h}^2(1 + \|\phi\|_{\Phi(X,\mathfrak{h})}^2). \quad (3.39)$$

3.6 Further properties of the T_ϕ semi-norm

Recall the definition of polynomial elements of \mathcal{N} in Definition 2.1. The following proposition bounds the T_ϕ semi-norm of a polynomial in terms of the T_0 semi-norm.

Proposition 3.10. *If F is a polynomial of degree $A \leq p_{\mathcal{N}}$ then*

$$\|F\|_{T_\phi} \leq \|F\|_{T_0} (1 + \|\phi\|_{\Phi})^A. \quad (3.40)$$

It is an immediate consequence of Proposition 3.10 that for $A \geq 0$ and any $\kappa \in (0, 2^{-1/2}]$,

$$\|F\|_{T_\phi} \leq \|F\|_{T_0} A^{A/2} \kappa^{-A} e^{\kappa^2 \|\phi\|_{\Phi}^2}. \quad (3.41)$$

For $A = 0$ this is trivial (with $0^0 = 1$), since then F is simply a complex number w and $\|F\|_{T_\phi} = \|F\|_{T_0} = |w|$. Also, for $A \geq 1$ and $\kappa \in (0, 2^{-1/2}]$, (3.41) follows from Proposition 3.10 together with the inequality

$$1 + x \leq \sqrt{2}(1 + x^2)^{1/2} \leq A^{1/2} \kappa^{-1} (1 + 2A^{-1} \kappa^2 x^2)^{1/2} \leq A^{1/2} \kappa^{-1} e^{A^{-1} \kappa^2 x^2}. \quad (3.42)$$

Suppose we have two test function spaces Φ and Φ' , with corresponding semi-norms T_ϕ and T'_ϕ . For $n \geq 0$, let

$$\rho^{(n)} = 2 \sup_{r \geq n} \sup_{g \in B(\Phi^{(r)})} \|g\|_{\Phi^{(r)}}. \quad (3.43)$$

In our applications, $\rho^{(n)}$ will be small for $n \geq 1$. The following proposition relates the T_ϕ and T'_ϕ semi-norms.

Proposition 3.11. *Let $A < p_N$ be a non-negative integer and let $F \in \mathcal{N}$. Then*

$$\|F\|_{T'_\phi} \leq (1 + \|\phi\|_{\Phi'})^{A+1} \left(\|F\|_{T'_0} + \rho^{(A+1)} \sup_{0 \leq t \leq 1} \|F\|_{T_{t\phi}} \right). \quad (3.44)$$

Recalling the discussion around (3.37), we can improve (3.44) by taking the infimum over all possible redefinitions of ϕ off X , with the result that

$$\|F\|_{T'_\phi} \leq (1 + \|\phi\|_{\Phi'(X)})^{A+1} \left(\|F\|_{T'_0} + \rho^{(A+1)} \sup_{0 \leq t \leq 1} \|F\|_{T_{t\phi}} \right) \quad \text{for } F \in \mathcal{N}(X). \quad (3.45)$$

Finally, the following proposition shows that the map θ of Definition 2.5 has a contractive property. For its statement, let Λ , Λ' and the map $z \mapsto z'$ be as described above Definition 2.5 and let $w : A \times \Lambda \rightarrow (0, \infty)$ and $w' : A \times \Lambda' \rightarrow (0, \infty)$ be weights as specified in Definition 3.1. These weights together define a new weight $w \sqcup w' : A \times (\Lambda \sqcup \Lambda') \rightarrow (0, \infty)$. Species in Λ and species in Λ' are distinct, and we order the species in such a way that a species from Λ' occurs immediately following its counterpart in Λ . We denote the corresponding norm on test functions $g : (\overrightarrow{\Lambda \sqcup \Lambda'})^* \rightarrow \mathbb{C}$ by $\Phi(w \sqcup w')$. Also, we define the function $w + w'$ from $A \times \Lambda$ to \mathbb{C} by

$$(w + w')(a, z) = w(a, z) + w'(a, z') \quad (3.46)$$

where $z \mapsto z'$ are the bijections between Λ and the copies Λ and Λ' .

Proposition 3.12. *For $F \in \mathcal{N}(\Lambda)$,*

$$\|\theta F\|_{T_{\phi \sqcup \xi}(w \sqcup w')} \leq \|F\|_{T_{\phi + \xi}(w + w')}. \quad (3.47)$$

Proofs of Propositions 3.10–3.12 are given in Sections 5.3–5.5 below.

3.7 Field regulators and associated norms

Definition 3.13. (a) The set $\Lambda = \mathbb{Z}^d/R^m$ is paved in a natural way by disjoint cubes of side R . We call these cubes *blocks* and denote the set of blocks by \mathcal{B} .

(b) A union of blocks is called a *polymer*, and the set of polymers is denoted \mathcal{P} . The size $|X|_R$ of $X \in \mathcal{P}$ is the number of blocks in X .

(c) A polymer X is *connected* if for any two points $x_a, x_b \in X$ there exists a path (x_0, \dots, x_n) in X with $\|x_{i+1} - x_i\|_\infty = 1$, $x_0 = x_a$ and $x_n = x_b$.

(d) A polymer $X \in \mathcal{P}$ is a *small set* if X is connected and $|X|_R \leq 2^d$. Let $\mathcal{S} \subset \mathcal{P}$ be the set of all

small sets.

(e) The *small set neighbourhood* of $X \subset \Lambda$ is the subset X^\square of Λ given by

$$X^\square = \bigcup_{Y \in \mathcal{S}: X \cap Y \neq \emptyset} Y. \quad (3.48)$$

(Other papers have used the notation X^* in place of X^\square , but we use X^\square to avoid confusion with our notation for sequence spaces.)

Note that, by definition, $X \subset X^\square$ and $(X \cup Y)^\square = X^\square \cup Y^\square$. The following definitions involve a positive parameter ℓ whose value will be chosen to satisfy the (related) hypotheses of Propositions 3.19–3.20 below. In these definitions, we consider only the case where $\Lambda_b = \Lambda \sqcup \bar{\Lambda}$ and the boson field is the complex field of Section 2.8.

Definition 3.14. Given $X \subset \Lambda$ and $\phi \in \mathbb{C}^\Lambda$, the *fluctuation-field regulator* is given by

$$G(X, \phi) = \prod_{x \in X} \exp \left(|B_x|^{-1} \|\phi\|_{\Phi(B_x^\square, \ell)}^2 \right), \quad (3.49)$$

where B_x is the unique block that contains x , and where the norm on the right-hand side is the $\Phi(\mathfrak{h})$ norm of Example 3.2 with $\mathfrak{h} = \ell > 0$ and localised to the small set neighbourhood B^\square as in (3.37). We define a norm on $\mathcal{N}(X^\square)$ by

$$\|F(X)\|_{G, \ell} = \sup_{\phi \in \Phi(\ell)} \frac{\|F(X)\|_{T_{\phi}(\ell)}}{G(X, \phi)} \quad \text{for } F(X) \in \mathcal{N}(X^\square). \quad (3.50)$$

Although the norm depends on X , we choose not to add a subscript X to the norm to make this dependence explicit.

For $X \in \mathcal{P}$ the formula (3.49) simplifies to

$$G(X, \phi) = \prod_{B \in \mathcal{B}(X)} \exp \|\phi\|_{\Phi(B^\square, \ell)}^2, \quad (3.51)$$

and the more complicated formula in the definition is a way to extend this simpler formula to all subsets $X \subset \Lambda$. A similar remark applies to the next definition.

Suppose that R and m are chosen in such a way that the diameter of B^\square is less than R^m (e.g., if m is sufficiently large). We can then identify B^\square with a subset of \mathbb{Z}^d and use this identification to define polynomial functions from B^\square to \mathbb{C} . The *dimension* of such a polynomial f , of a single variable, is defined to be $\frac{d-2}{2}$ plus the degree of f . Let $d_{\tilde{\Pi}}$ be a fixed non-negative integer. We define

$$\tilde{\Pi}(B^\square) = \{f \in \mathbb{C}^\Lambda \mid f \text{ restricted to } B^\square \text{ is a polynomial of dimension at most } d_{\tilde{\Pi}}\}. \quad (3.52)$$

Then, for $\phi \in \mathbb{C}^\Lambda$, we define the semi-norm

$$\|\phi\|_{\tilde{\Phi}(B^\square)} = \inf\{\|\phi - f\|_{\Phi} : f \in \tilde{\Pi}(B^\square)\}. \quad (3.53)$$

Definition 3.15. Given $X \subset \Lambda$ and $\phi \in \mathbb{C}^\Lambda$, the *large-field regulator* is given by

$$\tilde{G}(X, \phi) = \prod_{x \in X} \exp \left(\frac{1}{2} |B_x|^{-1} \|\phi\|_{\Phi(B_x^\square, \ell)}^2 \right). \quad (3.54)$$

The factor $\frac{1}{2}$, which does not occur in (3.49), has been inserted in (3.54) for later convenience. We define a norm on $\mathcal{N}(X^\square)$ by

$$\|F(X)\|_{\tilde{G}, \mathfrak{h}} = \sup_{\phi \in \Phi(\mathfrak{h})} \frac{\|F(X)\|_{T_\phi(\mathfrak{h})}}{\tilde{G}(X, \phi)} \quad \text{for } F(X) \in \mathcal{N}(X^\square), \quad (3.55)$$

where we have made explicit in the notation the fact that the norm on the left-hand side depends on a parameter \mathfrak{h} which may be chosen to be different from the parameter ℓ used for the regulators. The dependence of the norm on ℓ is left implicit.

It is immediate from the definitions that $G(X, \phi)$ and $\tilde{G}(X, \phi)$ are increasing in X , and that for all disjoint X, Y and for all $\phi \in \mathbb{C}^\Lambda$,

$$G(X \cup Y, \phi) = G(X, \phi)G(Y, \phi), \quad (3.56)$$

$$\tilde{G}(X \cup Y, \phi) = \tilde{G}(X, \phi)\tilde{G}(Y, \phi). \quad (3.57)$$

In addition, for $A \geq 0$ there is a $c_A \geq 1$ such that for all $t \in [0, 1]$,

$$\begin{aligned} 1 = G(X, 0) &\leq G(X, \phi), \quad \tilde{G}(X, t\phi) \leq G^{1/2}(X, \phi), \\ (1 + \|\phi\|_{\Phi(\ell, X^\square)})^{A+1} &\leq c_A G^{1/2}(X, \phi). \end{aligned} \quad (3.58)$$

The first two inequalities are valid for $X \subset \Lambda$. The third holds for $X \in \mathcal{P}$, and follows from (3.51). The following proposition extends the product property to the G and \tilde{G} norms.

Proposition 3.16. *If X, Y are disjoint and if $F(X) \in \mathcal{N}(X^\square)$, $i = 1, 2$ and $K(Y) \in \mathcal{N}(Y^\square)$, then $F(X)K(Y) \in \mathcal{N}((X \cup Y)^\square)$, and for either of the G or \tilde{G} norms (3.50) and (3.55),*

$$\|F(X)K(Y)\| \leq \|F(X)\| \|K(Y)\|. \quad (3.59)$$

Proof. This follows immediately from the product property Proposition 3.7 for the T_ϕ semi-norm, together with (3.56)–(3.57). \blacksquare

By definition,

$$\|F\|_{T_0(\ell)} \leq \|F\|_{G, \ell}. \quad (3.60)$$

The following proposition shows that this inequality can be partially reversed, at the expense of a term involving a multiple of $\|F\|_{\tilde{G}}$. In our application, the ratio ℓ/\mathfrak{h} appearing in this term will be small.

Proposition 3.17. *Let $X \in \mathcal{P}$ and $F \in \mathcal{N}(X)$. For any positive integer $A < p_N$, there is a constant c_A such that*

$$\|F\|_{G, \ell} \leq c_A \left(\|F\|_{T_0(\ell)} + \left(\frac{\ell}{\mathfrak{h}} \right)^{A+1} \|F\|_{\tilde{G}, \mathfrak{h}} \right). \quad (3.61)$$

Proof. We apply Proposition 3.11, with $T'_\phi = T_\phi(\ell)$ and $T_\phi = T_\phi(\mathfrak{h})$. Then $\rho^{(n)} = (\ell/\mathfrak{h})^n$ by definition. It follows from (3.55) and (3.58) that

$$\|F\|_{T_\phi} \leq \|F\|_{\tilde{G}, \mathfrak{h}} \tilde{G}(X, t\phi) \leq \|F\|_{\tilde{G}, \mathfrak{h}} G^{1/2}(X, \phi). \quad (3.62)$$

We use this in the last term on the right-hand side of (3.44), to obtain

$$\|F\|_{T_\phi(\ell)} \leq (1 + \|\phi\|_{\Phi(\ell)})^{A+1} \left(\|F\|_{T_0(\ell)} + \left(\frac{\ell}{\mathfrak{h}}\right)^{A+1} \|F\|_{\tilde{G}, \mathfrak{h}} G^{1/2}(X, \phi) \right). \quad (3.63)$$

We then apply (3.58), divide by $G(X, \phi)$, and take the supremum over ϕ to obtain (3.61). \blacksquare

3.8 Norm estimates for Gaussian integration

The following proposition shows that the Laplacian, and in view of (2.16) also the Gaussian integral, are bounded operators on a space of polynomials in \mathcal{N} . In its statement, we regard \mathbf{C} as a test function in Φ , by extending the definition above (2.13) to $\mathbf{C}_z = 0$ for $z \in \bar{\Lambda}^*$ unless the length of z is 2 and both components are either in Λ_b or in Λ_f , in which case it is given respectively by $\mathbf{C}_{b;z}$ or $\mathbf{C}_{f;z}$. Then it makes sense to take the norm $\|\mathbf{C}\|_\Phi$.

Proposition 3.18. *If $F \in \mathcal{N}$ is a polynomial of degree at most A , with $A \leq p_{\mathcal{N}}$, then*

$$\|\Delta_{\mathbf{C}} F\|_{T_\phi} \leq A^2 \|\mathbf{C}\|_\Phi \|F\|_{T_\phi} \quad (3.64)$$

and

$$\|e^{t\Delta_{\mathbf{C}}} F\| \leq e^{|t|A^2\|\mathbf{C}\|_\Phi} \|F\|_{T_\phi}. \quad (3.65)$$

Note that (3.65) follows from $\|e^{t\Delta_{\mathbf{C}}}\| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \|t\Delta_{\mathbf{C}}\|^n$ together with (3.64), so it suffices to prove (3.64).

In the next proposition, we restrict to the conjugate fermion field setting of Section 2.7, with fields $(\bar{\psi}_x, \psi_x)_{x \in \Lambda}$. We extend C_f to a test function in $\Phi(\Lambda)$ by setting it equal to zero when evaluated on any sequence z except those where z has length 2 and both components are in Λ . Then the norm $\|C_f\|_{\Phi(w')}$ makes sense.

Proposition 3.19. *In the conjugate fermion field setting of Section 2.7, suppose that the covariance C_f obeys $\|C_f\|_{\Phi(w')} \leq 1$. If $F \in \mathcal{N}(\Lambda \sqcup \Lambda')$ then*

$$\|\mathbb{E}_{\mathbf{C}} F\|_{T_\phi(w)} \leq \mathbb{E}_{\mathbf{C}_b} \|F\|_{T_{\phi \sqcup \xi}(w \sqcup w')}. \quad (3.66)$$

Also, if $F \in \mathcal{N}(\Lambda)$ then

$$\|\mathbb{E}_{\mathbf{C}} \theta F\|_{T_\phi(w)} \leq \mathbb{E}_{\mathbf{C}_b} \|\theta F\|_{T_{\phi \sqcup \xi}(w \sqcup w')} \leq \mathbb{E}_{\mathbf{C}_b} \|F\|_{T_{\phi+\xi}(w+w')}. \quad (3.67)$$

The variable ξ , which occurs in (3.67) (and also in (3.69)) is a dummy variable of integration for $\mathbb{E}_{\mathbf{C}_b}$. Note that the first inequality of (3.67) is an immediate consequence of (3.66), and that the second follows from (3.47), so it suffices to prove (3.66). In fact, as we show in Lemma 6.4 below, a stronger statement than (3.66) holds. Namely, if $h : \mathbb{R}^{\Lambda_b} \rightarrow \mathbb{C}$ then

$$\|\mathbb{E}_{\mathbf{C}} h F\|_{T_\phi(w)} \leq \mathbb{E}_{\mathbf{C}_b} [|h(\xi)| \|F\|_{T_{\phi \sqcup \xi}(w \sqcup w')}] . \quad (3.68)$$

Finally, we have the an estimate for the Gaussian expectation of the fluctuation-field regulator.

Proposition 3.20. *Let $t \geq 0$, $\alpha_G > 1$, and $X \subset \Lambda$. There exists a (small) positive constant $c(\alpha_G)$ such that if $\|\mathbf{C}_b\|_{\Phi+(\ell)} \leq c(\alpha_G)t^{-1}$, where the Φ^+ norm is the Φ norm with p_Φ replaced by $p_\Phi + d$, then*

$$0 \leq \mathbb{E}_{\mathbf{C}_b} G^t(X, \xi) \leq \alpha_G^{R^{-d}|X|}. \quad (3.69)$$

Proofs of Propositions 3.18–3.20 are given in Sections 6.1–6.3 below.

4 Gaussian integration and the heat equation

In this section, we prove Proposition 2.6. The proof uses integration by parts. For the purely bosonic case, it is straightforward to apply integration by parts to obtain

$$\mathbb{E}_{\mathbf{C}_b} \phi_x f = \sum_{y \in \Lambda_b} \mathbf{C}_{b;x,y} \mathbb{E}_{\mathbf{C}_b} \frac{\partial f}{\partial \phi_y}, \quad x \in \Lambda_b \quad (4.1)$$

where f is any smooth function such that both sides are integrable. The following lemma is a fermionic version of (4.1). Although it is standard (see, e.g., [13, Proposition 1.17]), we give the simple proof.

Lemma 4.1. *For $F \in \mathcal{N}(\Lambda)$ and $x \in \Lambda_f$,*

$$\mathbb{E}_{\mathbf{C}_f} \psi_x F = \sum_{y \in \Lambda_f} \mathbf{C}_{f;x,y} \mathbb{E}_{\mathbf{C}_f} i_y F. \quad (4.2)$$

Proof. By definition,

$$i_y S = \frac{1}{2} \sum_{v \in \Lambda_f} \mathbf{A}_{f;y,v} \psi_v - \frac{1}{2} \sum_{u \in \Lambda_f} \mathbf{A}_{f;u,y} \psi_u = \sum_{v \in \Lambda_f} \mathbf{A}_{f;y,v} \psi_v. \quad (4.3)$$

It suffices by linearity to consider F a product of generators, and since $i_y F$ cannot contain all generators as factors,

$$\int_{\Lambda_f} i_y F = 0. \quad (4.4)$$

By replacing F by $e^{-S} F$, we have

$$\int_{\Lambda_f} (i_y e^{-S}) F + \int_{\Lambda_f} e^{-S} (i_y F) = 0. \quad (4.5)$$

This is the same as

$$\int_{\Lambda_f} e^{-S} \left(- \sum_v \mathbf{A}_{f;y,v} \psi_v \right) F + \int_{\Lambda_f} e^{-S} (i_y F) = 0. \quad (4.6)$$

By applying the inverse of \mathbf{A}_f to both sides, we obtain the desired result. \blacksquare

The following lemma provides the expression in our context of the intimate link between Gaussian integration and the heat equation. In the purely bosonic context, this is a standard fact about Gaussian random variables.

Lemma 4.2. For $T > 0$ and $F \in \mathcal{N}(\Lambda)$ such that $F_t = \mathbb{E}_t \mathbf{C} \theta F$ is defined for $t < T$, the differential equation

$$\frac{d}{dt} F_t = \frac{1}{2} \Delta_{\mathbf{C}} F_t \quad (4.7)$$

holds for $t \in (0, T)$. Moreover, if $P \in \mathcal{N}(\Lambda)$ is a polynomial of finite degree, then

$$\mathbb{E}_{\mathbf{C}} \theta P = e^{\frac{1}{2} \Delta_{\mathbf{C}}} P. \quad (4.8)$$

Proof. Since the Gaussian expectation factors as in (2.11), to prove (4.7) it suffices to consider separately the cases where F is purely bosonic or purely fermionic.

We first prove (4.7) in the bosonic case, where $F = f$ is a smooth function of ϕ . The expectation is then a standard Gaussian integral, and by a change of variables we have

$$\frac{d}{dt} F_t(\phi) = \frac{d}{dt} \mathbb{E}_{\mathbf{C}_b} F(\phi + \sqrt{t} \xi) = \mathbb{E}_{\mathbf{C}_b} \sum_{x \in \Lambda_b} F_x(\phi + \sqrt{t} \xi) \frac{1}{2\sqrt{t}} \xi_x. \quad (4.9)$$

To differentiate under the expectation we need to know that the resulting integrand is integrable. To see this, we observe that since $t < T$ there exists $\epsilon > 0$ such that $F(\phi + \sqrt{t} \xi) \exp[\epsilon \sum_x \xi_x^2]$ is integrable. Now we apply the integration by parts identity (4.1), and the definition (2.13) of the Laplacian, to conclude that

$$\begin{aligned} \frac{d}{dt} F_t(\phi) &= \frac{1}{2} \mathbb{E}_{\mathbf{C}_b} \sum_{x, y \in \Lambda_b} \mathbf{C}_{b; x, y} F_{x, y}(\phi + \sqrt{t} \xi) \\ &= \frac{1}{2} \mathbb{E}_{\mathbf{C}_b} \Delta_{\mathbf{C}_b} F(\phi + \sqrt{t} \xi) = \frac{1}{2} \Delta_{\mathbf{C}_b} \mathbb{E}_{\mathbf{C}_b} F(\phi + \sqrt{t} \xi) = \frac{1}{2} \Delta_{\mathbf{C}_b} F_t(\phi). \end{aligned} \quad (4.10)$$

This proves the bosonic case of (4.7).

For the fermionic case, we can suppose that $F = \psi^y = \psi_{y_1} \cdots \psi_{y_k}$. We first note that

$$\frac{d}{dt} \theta_t \psi^y = \frac{d}{dt} \prod_{j=1}^k (\psi_{y_j} + t \psi_{y'_j}) = \sum_i (-1)^{i-1} \psi_{y'_i} \prod_{j \neq i} (\psi_{y_j} + t \psi_{y'_j}), \quad (4.11)$$

with the factors under the product maintaining their original order. By definition of i_x , this gives

$$\frac{d}{dt} \theta_t \psi^y = \sum_{x \in \Lambda_f} \psi_x \frac{1}{t} i_x (\theta_t \psi^y) = \sum_{x \in \Lambda_f} \psi_x \theta_t (i_x \psi^y), \quad (4.12)$$

where the sum extends to all $x \in \Lambda_f$ because terms with $x \neq y'_j$ for some j vanish. With Lemma 4.1, we then obtain

$$\begin{aligned} \frac{d}{dt} \mathbb{E}_{\mathbf{C}_f} \theta_t F &= \sum_{x \in \Lambda_f} \mathbb{E}_{\mathbf{C}_f} \psi_x \theta_t (i_x F) \\ &= \sum_{x, y \in \Lambda_f} \mathbf{C}_{f; x, y} \mathbb{E}_{\mathbf{C}_f} i_y \theta_t (i_x F) = \sum_{x, y \in \Lambda_f} \mathbf{C}_{f; x, y} \mathbb{E}_{\mathbf{C}_f} \theta_t (t i_y i_x F), \end{aligned} \quad (4.13)$$

which is the same as

$$\frac{1}{t} \frac{d}{dt} \mathbb{E}_{\mathbf{C}_f} \theta_t F = \Delta_{\mathbf{C}_f} \mathbb{E}_{\mathbf{C}_f} \theta_t F. \quad (4.14)$$

Writing $\frac{1}{t} \frac{d}{dt} = 2 \frac{d}{d(t^2)}$, and then replacing t^2 by t , we obtain

$$\frac{d}{dt} \mathbb{E}_{\mathbf{C}_f} \theta_{\sqrt{t}} F = \frac{1}{2} \Delta_{\mathbf{C}_f} \mathbb{E}_{\mathbf{C}_f} \theta_{\sqrt{t}} F. \quad (4.15)$$

It can be verified from the definitions that $\mathbb{E}_{\mathbf{C}_f} \theta_{\sqrt{t}} = \mathbb{E}_{t\mathbf{C}_f} \theta$, and the fermionic case of (4.7) follows.

Finally, suppose that F is a polynomial P of finite degree. By (4.7), each of P_t and $e^{\frac{t}{2}\Delta_{\mathbf{C}}} P$ solves the heat equation with the same initial data. The heat equation is a finite-dimensional linear system of ordinary differential equations because Λ is a finite set and thus $\Delta_{\mathbf{C}}$ is a linear operator acting on the finite-dimensional vector space of polynomials in ϕ and ψ . Therefore solutions for the heat equation are unique by the standard theory of linear systems, and (4.8) follows. ■

Proof of Proposition 2.6. Since (2.16) has been proven in (4.8), it suffices to prove (2.15).

By the first equality of (2.11), it suffices to verify (2.15) individually for $F = f$ and $F = \psi^y$. For $F = \psi^y$, (2.15) is an immediate consequence of (4.8). For $F = f$, the expectation is a standard Gaussian expectation. Since finite Borel measures are uniquely characterised by their Fourier transforms, it suffices to consider the case $f(\phi) = e^{i\phi \cdot \eta}$ for $\eta \in \mathbb{R}^{\Lambda_b}$. The Fourier transform of a Gaussian measure with covariance \mathbf{C}_b is $e^{-(\eta, \mathbf{C}_b \eta)}$. Thus, setting $\mathbf{C}_b = \mathbf{C}_{b,1} + \mathbf{C}_{b,2}$, we have

$$\mathbb{E}_{\mathbf{C}_b} \theta f = e^{i\phi \cdot \eta} e^{-(\eta, \mathbf{C}_b \eta)}, \quad (4.16)$$

and also

$$\mathbb{E}_{\mathbf{C}_{b,2}} \theta (\mathbb{E}_{\mathbf{C}_{b,2}} \theta f) = \mathbb{E}_{\mathbf{C}_{b,2}} \theta (e^{i\phi \cdot \eta} e^{-(\eta, \mathbf{C}_{b,1} \eta)}) = e^{i\phi \cdot \eta} e^{-(\eta, \mathbf{C}_{b,2} \eta)} e^{-(\eta, \mathbf{C}_{b,1} \eta)} \quad (4.17)$$

The above two right-hand sides are equal, and (2.15) follows in the bosonic case. This completes the proof. ■

5 The T_ϕ semi-norm

We now prove the five propositions stated in Sections 3.4–3.6: the product property of Proposition 3.7, the exponential norm estimate of Proposition 3.8, the polynomial norm estimate of Proposition 3.10, the change of norm estimate of Proposition 3.11, and the contractive bound for the map θ of Proposition 3.12. Many of the proofs follow the strategy of writing the T_ϕ semi-norm in terms of the pairing (3.16) that defines it, and then introducing an adjoint operation that transfers the desired statement into an estimate on test functions.

5.1 Proof of the product property

In this section, we prove the product property stated in Proposition 3.7. The proof proceeds by first establishing the product property for a more general algebra with semi-norm, and then noting that the product property of the T_ϕ norm follows as an instance.

Let \mathcal{H} be the algebra, generated by the fermion field, and over the ring of formal power series in indeterminates $(\xi_x)_{x \in \Lambda_b}$. An element $A \in \mathcal{H}$ has a unique representation

$$A = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} F_z \xi^{z_b} \psi^{z_f}, \quad (5.1)$$

where $z = (z_b, z_f)$, the coefficients F_z are complex valued, symmetric in the components of $z_b \in \vec{\Lambda}_b^*$, and antisymmetric in the components of $z_f \in \vec{\Lambda}_f^*$. Let \mathcal{F} be the set of *admissible* coefficients that obey these symmetry requirements. As vector spaces, \mathcal{H} and \mathcal{F} are isomorphic by the map $A \mapsto (F_z)_{z \in \vec{\Lambda}^*}$ implicitly defined by (5.1).

We use this isomorphism to transport the product from \mathcal{H} to a product on \mathcal{F} . Let

$$\eta^z = \xi^{z_b} \psi^{z_f}. \quad (5.2)$$

For $F', F'' \in \mathcal{F}$, we define $(F' \star F'')$ to be the unique element of \mathcal{F} such that

$$\sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} (F' \star F'')_z \eta^z = \left(\sum_{z' \in \vec{\Lambda}^*} \frac{1}{z'!} F'_{z'} \eta^{z'} \right) \left(\sum_{z'' \in \vec{\Lambda}^*} \frac{1}{z''!} F''_{z''} \eta^{z''} \right). \quad (5.3)$$

The vector space isomorphism between \mathcal{H} and \mathcal{F} implies the existence of $F' \star F''$, and with the \star product, \mathcal{F} becomes an algebra isomorphic to \mathcal{H} .

For a sequence $x = (x_1, x_2, \dots, x_p)$, we say that (x', x'') are *complementary with respect to x* if x' is a subsequence of x and x'' is the sequence obtained by removing x' from x . The pairs such that x' or x'' is the empty sequence are included. We denote by S_x the set of all pairs (x', x'') that are complementary with respect to x . There is an inverse relation: given sequences x' and x'' we define $x' \diamond x''$ to be the set of all x such that $(x', x'') \in S_x$. We extend this notation to $z \in \vec{\Lambda}^*$ by applying it to z species by species. For example, with just one boson and one fermion species, (z', z'') are complementary with respect to z if $(z'_b, z''_b) \in S_{z_b}$ and $(z'_f, z''_f) \in S_{z_f}$. We define S_z to be the set of all (z', z'') that are complementary with respect to z and we define $z' \diamond z''$ to be the set of all $z \in \vec{\Lambda}^*$ such that $(z', z'') \in S_z$. Recall that factorials and concatenation were defined species-wise in $\vec{\Lambda}^*$, in Section 3.2. Also, given z_f and a permutation \tilde{z}_f of its elements, we write $\text{sgn}(z_f, \tilde{z}_f)$ for the sign of the permutation.

Lemma 5.1. *Let $F' = \sum_{z'} \frac{1}{z'!} F'_{z'} \eta^{z'}$ and $F'' = \sum_{z''} \frac{1}{z''!} F''_{z''} \eta^{z''}$. The product defined on \mathcal{F} by (5.3) is given by*

$$(F' \star F'')_z = \sum_{(z', z'') \in S_z} F'_{z'} F''_{z''} \text{sgn}(z'_f \circ z''_f, z_f). \quad (5.4)$$

Proof. Let $(F' \star F'')_z$ denote the right-hand side of (5.4), and set $F' * F'' = \sum_z \frac{1}{z!} (F' \star F'')_z \eta^z$. It suffices to show that

$$F' * F'' \in \mathcal{F}, \quad (5.5)$$

and

$$F' * F'' = F' F'', \quad (5.6)$$

where the product on the right-hand side is the product in the algebra \mathcal{F} .

First, by definition,

$$\begin{aligned} (F' * F'')_z \eta^z &= \sum_{(z', z'') \in S_z} F'_{z'} F''_{z''} \operatorname{sgn}(z'_f \circ z''_f, z_f) \eta^z \\ &= \sum_{(z', z'') \in S_z} F'_{z'} F''_{z''} \eta^{z' \circ z''} = \sum_{(z', z'') \in S_z} F'_{z'} \eta^{z'} F''_{z''} \eta^{z''}. \end{aligned} \quad (5.7)$$

From (5.7), we obtain

$$\begin{aligned} F' * F'' &= \sum_z \frac{1}{z!} \sum_{(z, z') \in S_z} F'_{z'} \eta^{z'} F''_{z''} \eta^{z''} = \sum_{z', z''} \frac{1}{z'!} \frac{1}{z''!} F'_{z'} \eta^{z'} F''_{z''} \eta^{z''} \sum_{z \in z' \diamond z''} \frac{z'! z''!}{z!} \\ &= \sum_{z', z''} \frac{1}{z'!} \frac{1}{z''!} F'_{z'} \eta^{z'} F''_{z''} \eta^{z''} = F' F'', \end{aligned} \quad (5.8)$$

where in the third equality we used the fact that the number of z in $z' \diamond z''$ is $z!/(z'!z''!)$. This proves (5.6).

For $F : \vec{\Lambda}^* \rightarrow \mathbb{C}$, let $\tilde{F}_z = F_z \eta^z$. To prove (5.5), it suffices to show that $\tilde{F}_{\pi z}$ does not depend on the permutation π of z , since this is equivalent to the symmetry conditions which define admissibility. Given $(z', z'') \in S_{\pi z}$, we can define $(\hat{z}', \hat{z}'') \in S_z$ in a unique way by reordering the components of z' to produce \hat{z}' and similarly for z'' . Then, by (5.7), we obtain

$$(\widetilde{F' * F''})_{\pi z} = \sum_{(z', z'') \in S_{\pi z}} \tilde{F}'_{z'} \tilde{F}''_{z''} = \sum_{(z', z'') \in S_z} \tilde{F}'_{\hat{z}'} \tilde{F}''_{\hat{z}''} = \sum_{(z', z'') \in S_z} \tilde{F}'_{z'} \tilde{F}''_{z''} = (\widetilde{F' * F''})_z, \quad (5.9)$$

where the third equality holds since F' and F'' are elements of \mathcal{F} . This proves (5.5), and completes the proof. \blacksquare

Given $F \in \mathcal{F}$ and a test function $g \in \Phi$, we define a pairing and a semi-norm by

$$\langle F, g \rangle = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} F_z g_z, \quad \|F\|_T = \sup_{g \in B(\Phi)} |\langle F, g \rangle|. \quad (5.10)$$

The following proposition shows that the T semi-norm on \mathcal{F} obeys the product property.

Proposition 5.2. *For all $F, G \in \mathcal{F}$, $\|F \star G\|_T \leq \|F\|_T \|G\|_T$.*

Proof. Let $g \in \Phi$ and $G \in \mathcal{F}$. By Lemma 5.1,

$$\begin{aligned} \langle F \star G, g \rangle &= \sum_z \frac{1}{z!} \sum_{(z', z'') \in S_z} F_{z'} G_{z''} \operatorname{sgn}(z'_f \circ z''_f, z_f) g_z \\ &= \sum_{z', z''} \sum_{z \in z' \diamond z''} \frac{1}{z!} F_{z'} G_{z''} \operatorname{sgn}(z'_f \circ z''_f, z_f) g_z. \end{aligned} \quad (5.11)$$

We define $G^*g \in \Phi$ by

$$(G^*g)_{z'} = \sum_{z''} \frac{1}{z''!} G_{z''} \sum_{z \in z' \diamond z''} \frac{z'!z''!}{z!} \text{sgn}(z'_f \circ z''_f, z_f) g_z, \quad (5.12)$$

so that

$$\langle F \star G, g \rangle = \sum_{z'} \frac{1}{z'!} F_{z'} (G^*g)_{z'} = \langle F, G^*g \rangle \quad (5.13)$$

and hence

$$\|F \star G\|_T \leq \|F\|_T \sup_{g \in B(\Phi)} \|G^*g\|_\Phi. \quad (5.14)$$

Thus it remains to show that

$$\|G^*g\|_\Phi \leq \|G\|_T \quad \text{for } g \in B(\Phi). \quad (5.15)$$

Given $g \in B(\Phi)$ and $z' \in \vec{\Lambda}^*$, we define a test function $f_{z'} \in \Phi$ by

$$f_{z',z''} = \sum_{z \in z' \diamond z''} \frac{z'!z''!}{z!} \text{sgn}(z'_f \circ z''_f, z_f) g_z. \quad (5.16)$$

We regard this as a function of z'' with z' fixed. By definition, $(G^*g)_{z'} = \langle G, f_{z'} \rangle$, and hence, by Definition 3.1,

$$\|G^*g\|_\Phi = \sup_{(\alpha', z') \in \mathcal{A}'} |\lambda_{\alpha', z'} \langle G, f_{z'} \rangle| = \sup_{(\alpha', z') \in \mathcal{A}'} |\langle G, \lambda_{\alpha', z'} f_{z'} \rangle|, \quad (5.17)$$

where \mathcal{A}' denotes a copy of \mathcal{A} , and where we have made the abbreviation $\lambda_{\alpha', z'} = w_{\alpha', z'}^{-1} \nabla^{\alpha'}$. Thus we obtain

$$\|G^*g\|_\Phi \leq \|G\|_T \sup_{(\alpha', z') \in \mathcal{A}'} \|\lambda_{\alpha', z'} f_{z'}\|_\Phi. \quad (5.18)$$

Thus, it is sufficient to show that for all $g \in B(\Phi)$ and $(\alpha', z') \in \mathcal{A}'$, $(\alpha'', z'') \in \mathcal{A}''$,

$$|\lambda_{\alpha'', z''} \lambda_{\alpha', z'} f_{z', z''}| \leq 1. \quad (5.19)$$

In (5.16), the operations $\lambda_{\alpha'', z''} \lambda_{\alpha', z'}$ can be interchanged with the summation because they are linear, and with the factorials and sgn function since these depend only on the length and order of the relevant sequences. Since the number of terms in the sum over $z \in z' \diamond z''$ is equal to $z'!/(z'!z''!)$, we find after taking the absolute values inside the summation that it suffices to show that, for each $z \in z' \diamond z''$,

$$|\lambda_{\alpha'', z''} \lambda_{\alpha', z'} g_z| \leq 1, \quad (5.20)$$

where the derivatives within the λ factors act on the arguments of g_z according to their permuted locations within $z \in z' \diamond z''$. Since (5.20) is a consequence of $g \in B(\Phi)$ and the definition of the Φ norm, this completes the proof. \blacksquare

Proof of Proposition 3.7. Let $F = \sum_{y \in \Lambda_f^*} \frac{1}{y!} F_y \psi^y \in \mathcal{N}$. For boson fields ϕ, ξ , Taylor expansion of the coefficients F_y about a fixed ϕ in powers of ξ defines an algebra isomorphism

$$F \mapsto \sum_{(x,y) \in \Lambda^*} \frac{1}{x!y!} F_{x,y}(\phi) \xi^x \psi^y \quad (5.21)$$

of \mathcal{N} into a subalgebra (if $p_{\mathcal{N}} < \infty$) of the algebra \mathcal{H} and, in turn, \mathcal{H} is isomorphic as an algebra to \mathcal{F} . The composition of these isomorphisms is an isometry of the semi-normed algebras (\mathcal{N}, T_ϕ) and (\mathcal{F}, T) , so Proposition 3.7 follows from Proposition 5.2. \blacksquare

5.2 Exponential norm estimate

In this section, we prove Proposition 3.8.

Let $f(u) = \sum_{n=0}^{\infty} a_n u^n$ and $h(u) = \sum_{n=0}^{\infty} |a_n| u^n$, and let $\|\cdot\|$ denote any semi-norm that obeys the product property, e.g., the T_ϕ semi-norm. As an immediate consequence of the product property, for any F , we have

$$\|f(F)\| \leq \sum_{n=0}^{\infty} |a_n| \|F^n\| \leq \sum_{n=0}^{\infty} |a_n| \|F\|^n = h(\|F\|). \quad (5.22)$$

It follows from (5.22) that

$$\|e^{-F}\|_{T_\phi} \leq e^{\|F\|_{T_\phi}}. \quad (5.23)$$

Proposition 3.8 provides an improvement to (5.23) when the purely bosonic part of F has positive real part. Its proof is based on the following lemma.

Lemma 5.3. *Let $\|\cdot\|$ denote any semi-norm that obeys the product property. If $\|F\| \leq 1$, then*

$$\|e^{-\frac{1}{2}F^2} - (1 + F)e^{-F}\| \leq \|e^{-\frac{1}{2}F^2}\| \|F^3\|. \quad (5.24)$$

Proof. Let

$$R = e^{-\frac{1}{2}F^2} - (1 + F)e^{-F}. \quad (5.25)$$

Let

$$f(z) = 1 + (z - 1)e^{z + \frac{1}{2}z^2}. \quad (5.26)$$

Then $R = e^{-\frac{1}{2}F^2} f(-F)$. By definition, $f(0) = 0$ and $f'(z) = z^2 e^{z + \frac{1}{2}z^2}$. Thus $f'(z)$ has a power series with non-negative coefficients, and hence so does $f(z)$. Also, $f(z) = z^3 g(z)$, for some $g(z) = \sum b_n z^n$ with $b_n \geq 0$. In addition, $g(1) = f(1) = 1$. Therefore, by (5.22),

$$\|f(-F)\| = \|F^3 g(-F)\| \leq \|F^3\| \|g(-F)\| \leq \|F^3\| g(\|F\|). \quad (5.27)$$

If $\|F\| \leq 1$, this simplifies to

$$\|f(-F)\| \leq \|F^3\|. \quad (5.28)$$

This gives

$$\|R\| = \|e^{-\frac{1}{2}F^2} f(-F)\| \leq \|e^{-\frac{1}{2}F^2}\| \|F^3\|, \quad (5.29)$$

which is (5.24). \blacksquare

Proof of Proposition 3.8. Let $F \in \mathcal{N}$ and let $F_\emptyset(\phi)$ be the purely bosonic part of F . We will prove that

$$\|e^{-F}\|_{T_\phi} \leq e^{-2\operatorname{Re}F_\emptyset(\phi) + \|F\|_{T_\phi}}. \quad (5.30)$$

We first assume that $|F_\emptyset(\phi)|$ is sufficiently small that $|1 - F_\emptyset(\phi)| - |F_\emptyset(\phi)| \geq 0$, and we write $F_\emptyset(\phi) = z = x + iy$. We show that this implies that

$$|1 - z| - |z| \leq 1 - 2x, \quad (5.31)$$

as follows. By hypothesis, (5.31) is equivalent to the inequality obtained by squaring both sides, and algebra reduces the latter to

$$x(1 - x) + y^2 \leq |z| |1 - z|. \quad (5.32)$$

This certainly holds if the left-hand side is negative, and otherwise it suffices to show that the inequality is valid if both sides are squared, and the latter reduces to

$$2x(1 - x) \leq (1 - x)^2 + x^2, \quad (5.33)$$

which does hold. This completes the proof of (5.31) when $|1 - F_\emptyset(\phi)| - |F_\emptyset(\phi)| \geq 0$.

The T_ϕ semi-norm is defined via the pairing given in (3.16). Let g be any test function of norm at most 1. By separating out the null contribution to the sum over z we have

$$\begin{aligned} |\langle 1 - F, g \rangle_\phi| &\leq |(1 - F_\emptyset(\phi))g_\emptyset| + \sum_{r \neq 0} \left| \sum_{z \in \bar{\Lambda}^{(r)}} \frac{1}{z!} F_z(\phi) g_z \right| \\ &= (|1 - F_\emptyset(\phi)| - |F_\emptyset(\phi)|) |g_\emptyset| + \sum_{r \geq 0} \left| \sum_{z \in \bar{\Lambda}^{(r)}} \frac{1}{z!} F_z(\phi) g_z \right|. \end{aligned} \quad (5.34)$$

We take the supremum over test functions g of unit norm. The final term becomes $\|F\|_{T_\phi}$, so

$$\|1 - F\|_{T_\phi} \leq |1 - F_\emptyset(\phi)| - |F_\emptyset(\phi)| + \|F\|_{T_\phi}. \quad (5.35)$$

For the rest of the proof, we drop the T_ϕ subscript. Given ϕ , we choose N sufficiently large that $|1 - \frac{1}{N}F_\emptyset(\phi)| - \frac{1}{N}|F_\emptyset(\phi)| \geq 0$. By (5.35) and (5.31),

$$\left\| 1 - \frac{1}{N}F \right\| \leq 1 - \frac{2}{N}\operatorname{Re}F_\emptyset(\phi) + \frac{1}{N}\|F\|. \quad (5.36)$$

By the product property,

$$\left\| \left(1 - \frac{1}{N}F \right)^N \right\| \leq \left(1 - \frac{2}{N}\operatorname{Re}F_\emptyset(\phi) + \frac{1}{N}\|F\| \right)^N \leq e^{-2\operatorname{Re}F_\emptyset(\phi) + \|F\|}. \quad (5.37)$$

It suffices now to show that the limit $N \rightarrow \infty$ can be taken inside the semi-norm on the left-hand side. For this we define $A = e^{F/N}(1 - \frac{1}{N}F)$. By (5.22) with $f(z) = e^z$ and with $f(z) = e^z - 1$, we

have $\|e^{-\frac{1}{2}F^2/N^2}\| = O(1)$ and $\|e^{-\frac{1}{2}F^2/N^2} - 1\| = O(N^{-2})$ as $N \rightarrow \infty$. Therefore, by (5.24) with F replaced by $-F/N$,

$$\begin{aligned}\|A - 1\| &\leq \|e^{F/N}(1 - \frac{1}{N}F) - e^{-\frac{1}{2}F^2/N^2}\| + \|e^{-\frac{1}{2}F^2/N^2} - 1\| \\ &= O(N^{-3}) + O(N^{-2}) = O(N^{-2}).\end{aligned}\tag{5.38}$$

Now let $f(z) = (1 - z^N)(1 - z)^{-1} = \sum_{n=0}^{N-1} z^n$. Then, by (5.22),

$$\begin{aligned}\|(1 - \frac{1}{N}F)^N - e^{-F}\| &= \|e^{-F}(A^N - 1)\| \\ &\leq \|e^{-F}\| \|A - 1\| f(\|A\|) = O(N^{-2})f(\|A\|),\end{aligned}\tag{5.39}$$

and the right-hand side is $O(N^{-1})$ since $f(1 + O(N^{-2})) = O(N)$. This completes the proof. \blacksquare

5.3 Polynomial norm estimate

In this section, we prove Proposition 3.10. We begin with some definitions and a preliminary lemma which will be useful also in Sections 5.4–5.5.

For $z \in \vec{\Lambda}^*$, let B_z denote the set of pairs $(z', z_b'') \in \vec{\Lambda}^* \times \vec{\Lambda}_b^*$ such that $z' \circ z_b'' = z$. For $s \in [0, 1]$, $g \in \Phi$, $\xi \in \mathbb{R}^{\Lambda_b}$ and $z \in \vec{\Lambda}^*$, we define a new test function $\sigma^*(s)g \in \Phi$ by setting $(\sigma^*(s)g)_z = 0$ if the length of z exceeds p_N , and otherwise

$$(\sigma_\xi^*(s)g)_z = \sum_{(z', z_b'') \in B_z} \frac{z!}{z'!z_b''!} s^{z_b''} \xi^{z_b''} g_{z'}.\tag{5.40}$$

We write $\sigma_\xi^{*(m)}g$ to denote the m^{th} derivative of $\sigma_\xi^*(s)g$ at $s = 0$.

Lemma 5.4. *For $s, t \in [0, 1]$, $g \in \Phi$, $P \in \mathcal{N}$ a polynomial of degree at most p_N , and $\phi, \xi \in \mathbb{R}^{\Lambda_b}$,*

$$\langle P, g \rangle_{t\phi+s\xi} = \langle P, \sigma_\xi^*(s)g \rangle_{t\phi}.\tag{5.41}$$

If $g \in \Phi^{(p)}$ and $m + p \leq p_N$, then $\sigma_\xi^{(m)}g \in \Phi^{(m+p)}$, and, for any $F \in \mathcal{N}$,*

$$\frac{d^m}{dt^m} \langle F, g \rangle_{t\phi} = \langle F, \sigma_\phi^{*(m)}g \rangle_{t\phi}.\tag{5.42}$$

For all p and for $g \in \Phi$,

$$\|\sigma_\xi^*(1)g\|_{\Phi^{(p)}} \leq (1 + \|\xi\|_\Phi)^p \|g\|_\Phi,\tag{5.43}$$

and

$$\|\sigma_\xi^{*(m)}g\|_{\Phi^{(m+p)}} \leq \frac{(m+p)!}{p!} \|\xi\|_\Phi^m \|g\|_\Phi.\tag{5.44}$$

Proof. By definition, for $g \in \Phi$ and for a polynomial P of degree p_N ,

$$\begin{aligned} \langle P, g \rangle_{t\phi+s\xi} &= \sum_{z'} \frac{1}{z'!} P_{z'}(t\phi + s\xi) g_{z'} = \sum_{z', z''_b} \frac{1}{z'! z''_b!} P_{z' \circ z''_b}(t\phi) s^{z''_b} \xi^{z''_b} g_{z'} \\ &= \sum_z \frac{1}{z!} P_z(t\phi) \sum_{(z', z''_b) \in B(z)} \frac{z!}{z'! z''_b!} s^{z''_b} \xi^{z''_b} g_{z'} = \langle P, \sigma_\xi^*(s) g \rangle_{t\phi}, \end{aligned} \quad (5.45)$$

which proves (5.41). If $g \in \Phi^{(p)}$ then differentiation of (5.40) gives

$$(\sigma_\xi^{*(m)} g)_z = \mathbb{1}_{z' \circ z''_b = z} \frac{(m+p)!}{p!} \xi^{z''_b} g_{z'}, \quad (5.46)$$

so $\sigma_\xi^{*(m)} g \in \Phi^{(m+p)}$. Also, when $g \in \Phi^{(p)}$, we may regard F in (5.42) as a polynomial and thus by (5.41) we obtain (5.42) via differentiation with respect to s (with $\xi = \phi$).

By the triangle inequality and (3.13), for $p \leq p_N$,

$$\|\sigma_\xi^*(1)g\|_{\Phi^{(p)}} \leq \sum_{\substack{p', p''_b : \\ p' + p''_b = p}} \frac{p!}{p'! p''_b!} \|\xi\|_\Phi^{p''_b} \|g\|_\Phi = (1 + \|\xi\|_\Phi)^p \|g\|_\Phi. \quad (5.47)$$

Since the left-hand side is zero for $p > p_N$, this proves (5.43). For (5.44), we only consider the case $m+p \leq p_N$ because otherwise the left-hand side is zero. Also we can assume that $g \in \Phi^{(p)}$ because no other part of g can contribute to the left-hand side. If $g \in \Phi^{(p)}$ and $m+p \leq p_N$, then from (5.46) and (3.13) we have

$$\|\sigma_\xi^{*(m)} g\|_{\Phi^{(m+p)}} \leq \frac{(m+p)!}{p!} \|\xi\|_\Phi^m \|g\|_\Phi. \quad (5.48)$$

This proves (5.44), and completes the proof. ■

Remark 5.5. It follows from Lemma 5.4 that for $F \in \mathcal{N}$, $g \in \Phi^{(p)}$, and for $m+p \leq p_N$,

$$\left| \frac{d^m}{dt^m} \langle F, g \rangle_{T_\phi + t\xi} \right| \leq \frac{(m+p)!}{m!} \|F\|_{T_\phi + t\xi(\Phi)} \|\xi\|_\Phi^m \|g\|_\Phi. \quad (5.49)$$

To see this, note that as in the proof of Lemma 5.4,

$$\left. \frac{d^m}{ds^m} \right|_0 \langle F, g \rangle_{T_\phi + s\xi(\Phi)} = \langle F, \sigma_\xi^{*(m)} g \rangle_{t\phi}. \quad (5.50)$$

With (5.44), this gives

$$\left| \left. \frac{d^m}{ds^m} \right|_0 \langle F, g \rangle_{T_\phi + s\xi(\Phi)} \right| \leq \frac{(m+p)!}{m!} \|F\|_{T_\phi} \|\xi\|_\Phi^m \|g\|_\Phi, \quad (5.51)$$

and then (5.49) follows by replacing ϕ with $\phi + t\xi$.

For $F \in \mathcal{N}$ and $t \geq 0$, we define $\sigma_t F \in \mathcal{N}$ by replacing the fields (ϕ, ψ) in F by $(t\phi, t\psi)$. For a positive integer A , $t \in [0, 1]$ and $F \in \mathcal{N}$, we define the truncated Taylor expansion for $\tau_t F$ by

$$\tau_t^{(\leq A)} F = \sum_{n=0}^A \frac{t^n}{n!} \tau_0^{(n)} F, \quad (5.52)$$

where $\tau_t^{(n)} F$ is the n^{th} derivative of $\tau_t F$ with respect to t . The following lemma gives the result of Proposition 3.10.

Lemma 5.6. *For $F \in \mathcal{N}$ and $A \leq p_{\mathcal{N}}$, let $P = \tau_1^{(\leq A)} F$. Then*

$$\|P\|_{T_\phi} \leq \|F\|_{T_0} (1 + \|\phi\|_\Phi)^A, \quad (5.53)$$

and if F is a polynomial of degree A then

$$\|F\|_{T_\phi} \leq \|F\|_{T_0} (1 + \|\phi\|_\Phi)^A. \quad (5.54)$$

Proof. The second claim is a consequence of the first because, in this case, $F = \tau_1^{(\leq A)} F$ by the uniqueness of Taylor expansions. To prove (5.53), we apply Lemma 5.4 with $\xi = \phi$, $t = 0$ and $s = 1$ to obtain

$$|\langle P, g \rangle_\phi| = |\langle P, \sigma_\phi^*(1)g \rangle_0| \leq \|P\|_{T_0} \|\sigma_\phi^*(1)g\|_\Phi. \quad (5.55)$$

Since $\|P\|_{T_0}$ is a truncation of the sum of positive terms that constitute $\|F\|_{T_0}$, it is the case that $\|P\|_{T_0} \leq \|F\|_{T_0}$. Also, we need only consider the case where $\sigma_\phi^*(1)g$ depends on at most A variables, since otherwise its pairing with P vanishes. It then follows from Lemma 5.4 with $\xi = \phi$ that

$$|\langle P, g \rangle_\phi| \leq \|F\|_{T_0} (1 + \|\phi\|_\Phi)^A \|g\|_\Phi. \quad (5.56)$$

Taking the supremum now over $g \in B(\Phi)$, we obtain (5.53) and the proof is complete. \blacksquare

5.4 Estimate with change of norm

The following lemma gives the result of Proposition 3.11.

Lemma 5.7. *Let $A < p_{\mathcal{N}}$ be a non-negative integer. For $F \in \mathcal{N}$, let $P = \tau_1^{(\leq A)} F$. Then*

$$\begin{aligned} \|F\|_{T'_\phi} &\leq (1 + \|\phi\|_{\Phi'})^{A+1} \left(\|P\|_{T'_0} + \rho^{(A+1)} \sup_{t \in [0,1]} \|F\|_{T_{t\phi}} \right) \\ &\leq (1 + \|\phi\|_{\Phi'})^{A+1} \left(\|F\|_{T'_0} + \rho^{(A+1)} \sup_{t \in [0,1]} \|F\|_{T_{t\phi}} \right). \end{aligned} \quad (5.57)$$

Proof. The second estimate follows from the first and $\|P\|_{T'_0} \leq \|F\|_{T'_0}$. To prove the first inequality let $R = F - P$. By the triangle inequality and Lemma 5.6 it is sufficient to prove that

$$\|R\|_{T'_\phi} \leq \rho^{(A+1)} (1 + \|\phi\|_{\Phi'})^{A+1} \sup_{t \in [0,1]} \|F\|_{T_{t\phi}}. \quad (5.58)$$

For this, it suffices to show that for a test function $g \in \Phi$ we have

$$|\langle R, g \rangle_\phi| \leq \rho^{(A+1)} (1 + \|\phi\|_{\Phi'})^{A+1} \sup_{t \in [0,1]} \|F\|_{T_{t\phi}} \|g\|_{\Phi'}. \quad (5.59)$$

We consider separately the cases (i) $g_z = 0$ for z with $p = p(z) \leq A$, and (ii) $g_z = 0$ except when $p = p(z) = 0, 1, \dots, A$. Any g can be decomposed into these two cases using $g = g\mathbb{1}_{p>A} + \sum_{r \leq A} g\mathbb{1}_{p=r}$, and

$$|\langle R, g \rangle_\phi| \leq |\langle R, g\mathbb{1}_{p>A} \rangle_\phi| + \sum_{r \leq A} |\langle R, g\mathbb{1}_{p=r} \rangle_\phi|. \quad (5.60)$$

For case (i), we simply note from (3.43) that

$$|\langle R, g \rangle_\phi| = |\langle F, g \rangle_\phi| \leq \|F\|_{T_\phi} \|g\|_\Phi \leq \|F\|_{T_\phi} \frac{1}{2} \rho^{(A+1)} \|g\|_{\Phi'}. \quad (5.61)$$

Note that the above right-hand side is at most half the right-hand side of (5.59).

For the more substantial case (ii), fix $g \in \Phi$ with $g_z = 0$ supported on sequences of length exactly p with some $p \in \{0, 1, \dots, A\}$. Let $f(t) = \langle R, g \rangle_{t\phi}$. By the Taylor remainder formula, for any $m \leq A + 1$,

$$|\langle R, g \rangle_\phi| \leq \frac{1}{m!} \sup_{t \in [0,1]} |f^{(m)}(t)|. \quad (5.62)$$

By Lemma 5.4 with $\xi = \phi$,

$$f^{(m)}(t) = \langle R, \sigma_\phi^{*(m)} g \rangle_{t\phi}. \quad (5.63)$$

Let $m = A + 1 - p$. Then we can replace $R = F - P$ by F in (5.63) because $\sigma_\phi^{*(m)} g$ is supported on sequences of length $m + p = A + 1$ by Lemma 5.4, whereas P is a polynomial of degree A . Therefore,

$$|\langle R, g \rangle_\phi| \leq \frac{1}{m!} \sup_{t \in [0,1]} \|F\|_{T_{t\phi}} \|\sigma_\phi^{*(m)} g\|_{\Phi^{(A+1)}}. \quad (5.64)$$

Since $\sigma_\phi^{*(m)} g$ is supported on sequences of length $A + 1$, by (3.43) we have

$$\|\sigma_\phi^{*(m)} g\|_{\Phi^{(A+1)}} \leq \frac{1}{2} \rho^{(A+1)} \|\sigma_\phi^{*(m)} g\|_{\Phi'^{(A+1)}}. \quad (5.65)$$

It follows from (5.64)–(5.65) and Lemma 5.4 that

$$\begin{aligned} |\langle R, g \rangle_\phi| &\leq \frac{1}{2} \rho^{(A+1)} \sum_{p=0}^A \binom{A+1}{p} \|\phi\|_{\Phi'}^{A+1-p} \|g\|_{\Phi'} \sup_{t \in [0,1]} \|F\|_{T_{t\phi}} \\ &\leq \frac{1}{2} \rho^{(A+1)} (1 + \|\phi\|_{\Phi'})^{A+1} \|g\|_{\Phi'} \sup_{t \in [0,1]} \|F\|_{T_{t\phi}}. \end{aligned} \quad (5.66)$$

Combined with the estimate for case (i), this gives (5.59) and completes the proof. \blacksquare

5.5 Contractive bound on θ

In this section, we prove Proposition 3.12.

Recall from the discussion above (2.14) that there is a bijection between a subset of Λ and Λ' , written $x \mapsto x'$. Recall from the discussion above Proposition 3.12 that species in Λ and species in Λ' are distinct, and are ordered in such a way that a species from Λ' occurs immediately following its counterpart in Λ . The *forget* function $f : \Lambda \sqcup \Lambda' \rightarrow \Lambda$ is defined by setting $f(x') = x$ when $x' \in \Lambda'$ and $f(x) = x$ when $x \in \Lambda$. We extend f to a map from $(\overrightarrow{\Lambda \sqcup \Lambda'})^*$ to $\vec{\Lambda}^*$ by letting f act componentwise on sequences. We define a map $\theta^* : \Phi(\Lambda \sqcup \Lambda') \rightarrow \Phi(\Lambda)$ by setting

$$(\theta^*g)_z = \sum_{\substack{v \in (\overrightarrow{\Lambda \sqcup \Lambda'})^* : \\ f(v) = z}} \frac{z!}{v!} g_v. \quad (5.67)$$

By definition, the $v!$ appearing in the above equation is equal to $u!u'!$, where u and u' are respectively the subsequences of v drawn from Λ and Λ' .

Lemma 5.8. For $F \in \mathcal{N}(\Lambda)$, $g \in \Phi(\Lambda \sqcup \Lambda')$, $\phi \in \mathbb{R}^{\Lambda_b}$ and $\xi \in \mathbb{R}^{\Lambda'_b}$,

$$\langle \theta F, g \rangle_{\phi \sqcup \xi} = \langle F, \theta^* g \rangle_{\phi + \xi}. \quad (5.68)$$

Proof. First, we compute the coefficients $(\theta F)_v$ for $v \in (\overrightarrow{\Lambda \sqcup \Lambda'})^*$, which is what is relevant for the pairing of θF with g . By Definition 2.5,

$$\theta F = \sum_{z_f \in \vec{\Lambda}_f^*} \frac{1}{z_f!} F_{z_f}(\phi + \xi)(\psi + \psi')^{z_f}. \quad (5.69)$$

We expand $F_{z_f}((\phi + \xi) + (\hat{\phi} + \hat{\xi}))$ in a power series in $\hat{\phi} + \hat{\xi}$ to obtain

$$\theta F = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} F_z(\phi + \xi)(\hat{\phi} + \hat{\xi})^{z_b}(\psi + \psi')^{z_f}. \quad (5.70)$$

Now we expand the binomials on the right-hand side and reorder the species within both the bosonic and fermionic products. We reorder the subscript on F_z in exactly the same way; then no sign change occurs. From this, we can read off the coefficients

$$(\theta F)_v = F_{f(v)}(\phi + \xi). \quad (5.71)$$

We abbreviate the right-hand side as $F_{f(v)} = F_{f(v)}(\phi + \xi)$. Then

$$\langle \theta F, g \rangle_{\phi \sqcup \xi} = \sum_{v \in (\overrightarrow{\Lambda \sqcup \Lambda'})^*} \frac{1}{v!} F_{f(v)} g_v = \sum_{z \in \vec{\Lambda}^*} \frac{1}{z!} F_z \sum_{v: f(v)=z} \frac{z!}{v!} g_v = \langle F, \theta^* g \rangle_{\phi + \xi}, \quad (5.72)$$

and the proof is complete. ■

Lemma 5.9. *The map $\theta^* : \Phi(w \sqcup w') \rightarrow \Phi(w + w')$ is a contraction, namely, for $g \in \Phi(w \sqcup w')$,*

$$\|\theta^* g\|_{\Phi(w+w')} \leq \|g\|_{\Phi(w \sqcup w')}. \quad (5.73)$$

Proof. In the following, $v \in (\overrightarrow{\Lambda \sqcup \Lambda'})^*$ and $z \in \vec{\Lambda}^*$. By (5.67),

$$\begin{aligned} |(w + w')_{\alpha, z}^{-1} (\nabla^\alpha \theta^* g)_z| &\leq (w + w')_{\alpha, z}^{-1} \sum_{v: f(v)=z} \frac{z!}{v!} |\nabla^\alpha g_v| \\ &\leq \|g\|_{\Phi(w \sqcup w')} (w + w')_{\alpha, z}^{-1} \sum_{v: f(v)=z} \frac{z!}{v!} (w \sqcup w')_{\alpha, v}. \end{aligned} \quad (5.74)$$

The final sum equals $(w + w')_{\alpha, z}$ by the binomial theorem; to see this we recall that v has species segregated so that in particular primed and unprimed variables are not interleaved, and the binomial coefficient $z!/v!$ accounts for the number of ways to desegregate these variables. Then (5.73) follows by taking the supremum over $(\alpha, z) \in \mathcal{A}$. \blacksquare

Proof of Proposition 3.12. Let $g \in B(\Phi(w \sqcup w'))$. By Lemma 5.8,

$$|\langle \theta F, g \rangle_{\phi \sqcup \xi}| = |\langle F, \theta^* g \rangle_{\phi + \xi}| \leq \|F\|_{T_{\phi + \xi}(w + w')} \|\theta^* g\|_{\Phi(w + w')}. \quad (5.75)$$

Taking the supremum over $g \in B(\Phi(w \sqcup w'))$ and applying Lemma 5.9, we have

$$\|\theta F\|_{T_{\phi \sqcup \xi}(w \sqcup w')} \leq \|F\|_{T_{\phi + \xi}(w + w')}. \quad (5.76)$$

This proves (3.47). \blacksquare

6 Integration norm estimates

In this section, we prove Propositions 3.18, 3.19 and 3.20.

6.1 Laplacian norm estimates

In this section, we prove Proposition 3.18. For this, it suffices to prove the following lemma, which slightly improves (3.64) by reducing the factor A^2 to $A(A - 1)$ on its right-hand side.

Lemma 6.1. *If $F \in \mathcal{N}$ is a polynomial of degree at most A , with $A \leq p_N$, then*

$$\frac{1}{2} \|\Delta_C F\|_{T_\phi} \leq \binom{A}{2} \|C\|_\Phi \|F\|_{T_\phi}. \quad (6.1)$$

Proof. For $g \in \Phi$ and $v \in \vec{\Lambda}^*$, let

$$(C^* g)_v = \mathbb{1}_{u \circ z = v} \frac{v!}{u! z!} C_u g_z \quad (6.2)$$

if the length of v is at most A , and otherwise $(\mathbf{C}^*g)_v = 0$. Here u denotes the first two coordinates of v and z denotes the others; in particular $(\mathbf{C}^*g)_v = 0$ if the length of v is less than 2. Then, by the definition of the Laplacian in (2.13),

$$\begin{aligned} \frac{1}{2} \langle \Delta_{\mathbf{C}} F, g \rangle_{\phi} &= \frac{1}{2} \sum_z \frac{1}{z!} (\Delta_{\mathbf{C}} F(\phi))_z g_z = \sum_{u,z} \frac{1}{u!z!} \mathbf{C}_u F_{u \circ z}(\phi) g_z \\ &= \sum_v \frac{1}{v!} F_v(\phi) (\mathbf{C}^*g)_v = \langle F, \mathbf{C}^*g \rangle_{\phi}. \end{aligned} \quad (6.3)$$

Since F is a polynomial of degree at most A , $F_v = 0$ as soon as the length of v exceeds A ; the fact that $A \leq p_N$ has been used in the last equality.

The binomial coefficient in (6.2) is at most $\binom{A}{2}$. With (3.13), this gives

$$\|\mathbf{C}^*g\|_{\Phi} \leq \binom{A}{2} \|\mathbf{C}\|_{\Phi} \|g\|_{\Phi} \quad (6.4)$$

and hence

$$\frac{1}{2} |\langle \Delta_{\mathbf{C}} F, g \rangle_{\phi}| \leq \|F\|_{T_{\phi}} \|\mathbf{C}^*g\|_{\Phi} \leq \|F\|_{T_{\phi}} \binom{A}{2} \|\mathbf{C}\|_{\Phi} \|g\|_{\Phi}, \quad (6.5)$$

and (6.1) follows by taking the supremum over $g \in B(\Phi)$. \blacksquare

6.2 The main integration estimate

In this section, we prove Proposition 3.19. For this, we adopt the conjugate fermion fields setting described in Section 2.7, with fields $\psi, \bar{\psi}$. As noted below the statement of Proposition 3.19, it suffices to prove the bound (3.66). The proof is based on the following lemma, which is known as *Gram's inequality*. A proof of Lemma 6.2 can be found in [13, Lemma 1.33].

Lemma 6.2. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. If $u_i, v_i \in H$ for $i = 1, \dots, n$, then*

$$\left| \det (\langle u_i, v_j \rangle)_{1 \leq i, j \leq n} \right| \leq \prod_{i=1}^n \langle u_i, u_i \rangle^{1/2} \langle v_i, v_i \rangle^{1/2}. \quad (6.6)$$

Recall that C_f can be interpreted as a test function as described above the statement of Proposition 3.19. Let E be the test function defined by $E_z = \mathbb{E}_{C_f} \psi^z$ for $z \in \bar{\Lambda}_f^*$, with the convention $E_{\emptyset} = 1$. For $z \in \bar{\Lambda}^* \setminus \bar{\Lambda}_f^*$ we set $E_z = 0$.

Lemma 6.3. *If $\|C_f\|_{\Phi} \leq 1$ then $\|E\|_{\Phi} \leq 1$.*

Proof. To simplify the notation, we drop the subscript f from C_f . By (2.20), we may assume that ψ^z has the form $\bar{\psi}_{x_1} \psi_{y_1} \cdots \bar{\psi}_{x_p} \psi_{y_p}$, in which case

$$E_z = \det C_{x,y}. \quad (6.7)$$

Let $\lambda_{\alpha,z} = (\prod_{i=1}^p \lambda_{\alpha'_i, x_i})(\prod_{i=1}^p \lambda_{\alpha''_i, y_i})$ with $\lambda_{\alpha'_i, x_i} = w_{\alpha'_i, x_i}^{-1} \nabla^{\alpha'_i}$ and $\lambda_{\alpha''_i, y_i} = w_{\alpha''_i, y_i}^{-1} \nabla^{\alpha''_i}$, with $\nabla^{\alpha'_i}$ acting on the x_i variable and $\nabla^{\alpha''_i}$ acting on the y_i variable. It suffices to prove that

$$|\lambda_{\alpha,z} E_z| \leq \prod_{i=1}^p (\lambda_{\alpha'_i, u} \lambda_{\alpha'_i, v} C_{u,v}|_{u=v=x})^{1/2} (\lambda_{\alpha''_i, u} \lambda_{\alpha''_i, v} C_{u,v}|_{u=v=y_i})^{1/2}, \quad (6.8)$$

since (6.8) implies the inequality

$$\|E\|_{\Phi} \leq \sup_{p \geq 1} \|C\|_{\Phi}^p. \quad (6.9)$$

By (6.7) and the fact the determinant is linear in rows and columns,

$$|\lambda_{\alpha,z} E_z| = |\lambda_{\alpha,z} \det C_{x,y}| = |\det(\lambda_{\alpha,z} C_{x,y})|. \quad (6.10)$$

We rewrite the determinant as follows. Let V be the vector space of all functions $f : \Lambda \rightarrow \mathbb{C}$. Given functions $h, k \in V$, we define

$$(h, k) = \sum_{x \in \Lambda} h_x k_x. \quad (6.11)$$

Then we define $f_i, g_i \in V$ by

$$(\lambda_{\alpha'_i, x_i} k)_{x_i} = (\delta_{x_i}, \lambda_{\alpha'_i, x_i} k) = (\lambda_{\alpha'_i, x_i}^{\dagger} \delta_{x_i}, k) = (f_i, k) \quad (6.12)$$

and

$$(\lambda_{\alpha''_i, y_i} h)_{y_i} = (\lambda_{\alpha''_i, y_i} h, \delta_{y_i}) = (h, \lambda_{\alpha''_i, y_i}^{\dagger} \delta_{y_i}) = (h, g_i). \quad (6.13)$$

We define an inner product on V by

$$\langle f, g \rangle = \sum_{x \in \Lambda} f_x C_{x,y} \bar{g}_y. \quad (6.14)$$

By definition, for $i, j \in \{1, \dots, p\}$,

$$\lambda_{\alpha'_i, x_i} \lambda_{\alpha''_j, y_j} C_{x_i, y_j} = \langle f_i, g_j \rangle, \quad (6.15)$$

and thus $\det(\lambda_{\alpha,z} C_{x,y}) = \det(\langle f_i, g_j \rangle)$. By Lemma 6.2,

$$|\det(\lambda_{\alpha,z} C_{x,y})| = |\det(\langle f_i, g_j \rangle)| \leq \prod_{i=1}^n \langle f_i, f_i \rangle^{1/2} \langle g_i, g_i \rangle^{1/2}. \quad (6.16)$$

For the right-hand side, we use

$$\langle f_i, f_i \rangle = \langle \lambda_{\alpha'_i, x_i}^{\dagger} \delta_{x_i}, \lambda_{\alpha'_i, x_i}^{\dagger} \delta_{x_i} \rangle = \lambda_{\alpha'_i, u} \lambda_{\alpha'_i, v} C_{u,v}|_{u=v=x_i}, \quad (6.17)$$

and similarly for $\langle g_i, g_i \rangle$. With (6.10), this proves (6.8) and completes the proof. \blacksquare

Proposition 3.19 is a consequence of the following lemma (with $h = 1$), which establishes (3.66).

Lemma 6.4. *In the conjugate fermion field setting of Section 2.7, suppose that the covariance satisfies $\|C_f\|_{\Phi(w')} \leq 1$. If $F \in \mathcal{N}(\Lambda \sqcup \Lambda')$ and $h : \mathbb{R}^{\Lambda'_b} \rightarrow \mathbb{C}$, then*

$$\|\mathbb{E}_{\mathbf{C}} hF\|_{T_\phi(w)} \leq \mathbb{E}_{\mathbf{C}_b} [|h(\xi)| \|F\|_{T_{\phi \sqcup \xi}(w \sqcup w')}] . \quad (6.18)$$

Proof. By definition, we can write $F = \sum_{z_f \in (\overrightarrow{\Lambda_f \sqcup \Lambda'_f})^*} \frac{1}{z_f!} F_{z_f} \psi^{z_f}$, with $F_{z_f} = F_{z_f}(\phi \sqcup \xi)$. Given z_f , let y be the subsequence of z_f such that $y \in \Lambda_f^*$, and let y' be the complementary subsequence of components of z_f in Λ'_f . The operator $\mathbb{E}_{\mathbf{C}}$ acts only on the ξ and ψ' variables. In particular,

$$\begin{aligned} \mathbb{E}_{\mathbf{C}_f} \psi^{z_f} &= \text{sgn}(z_f, y \circ y') \mathbb{E}_{\mathbf{C}_f} \psi^{y \circ y'} \\ &= \text{sgn}(z_f, y \circ y') \psi^y \mathbb{E}_{\mathbf{C}_f} \psi^{y'} = \text{sgn}(z_f, y \circ y') E_{y'} \psi^y, \end{aligned} \quad (6.19)$$

where on the right-hand side E denotes the test function of Lemma 6.3. Therefore, by (2.11),

$$\mathbb{E}_{\mathbf{C}} hF = \sum_{z_f \in (\overrightarrow{\Lambda_f \sqcup \Lambda'_f})^*} \frac{1}{z_f!} (\mathbb{E}_{\mathbf{C}_b} hF_{z_f}) \text{sgn}(z_f, y \circ y') E_{y'} \psi^y. \quad (6.20)$$

For $g \in \Phi(w)$, we define $E^*g \in \Phi(w \sqcup w')$ by

$$(E^*g)_z = \begin{cases} \text{sgn}(z_f, y \circ y') E_{y'} g_{z_b \circ y} & z_b \in \Lambda_b^* \\ 0 & z_b \notin \Lambda_b^* \end{cases} \quad (6.21)$$

for $z \in (\overrightarrow{\Lambda \sqcup \Lambda'})^*$. Then

$$\langle \mathbb{E}_{\mathbf{C}} hF, g \rangle_\phi = \sum_{z \in (\overrightarrow{\Lambda \sqcup \Lambda'})^*} \frac{1}{z!} (\mathbb{E}_{\mathbf{C}_b} hF_z) (E^*g)_z = \mathbb{E}_{\mathbf{C}_b} [h(\xi) \langle F, E^*g \rangle_{\phi \sqcup \xi}], \quad (6.22)$$

and hence

$$\begin{aligned} |\langle \mathbb{E}_{\mathbf{C}_b} hF, g \rangle_\phi| &\leq \mathbb{E}_{\mathbf{C}_b} [|h(\xi)| |\langle F, E^*g \rangle_{\phi \sqcup \xi}|] \\ &\leq (\mathbb{E}_{\mathbf{C}_b} [|h(\xi)| \|F\|_{T_{\phi \sqcup \xi}(w \sqcup w')}]) \|E^*g\|_{\Phi(w \sqcup w')}. \end{aligned} \quad (6.23)$$

Derivative operators ∇^α do not act on the sgn function, so we may apply (3.14) and then Lemma 6.3 to conclude that $\|E^*g\|_{\Phi(w \sqcup w')} \leq \|g\|_{\Phi(w)}$. Then (6.18) follows by taking the supremum over $g \in B(\Phi(w))$ in (6.23), and the proof is complete. \blacksquare

Since $(E^*g)_z$ vanishes by definition whenever z contains an entry in Λ'_b , the above proof shows that (6.18) could be strengthened by replacing the semi-norm on the right-hand side by the smaller semi-norm which does not involve derivatives with respect to the boson fluctuation field ξ .

6.3 Expectation of the fluctuation-field regulator

The main result of this section is Lemma 6.8, which immediately gives Proposition 3.20. In preparation for Lemma 6.8, we prove three preliminary lemmas. The first of these is proved in [4, Lemma 6.28], and a precursor of the second is [5, Lemma B.2].

Lemma 6.5. *Let $(\xi_a)_{a \in \mathcal{H}}$ be a finite set of Gaussian random variables with covariance C . Suppose that the largest eigenvalue of C is less than $\frac{1}{2}$. Let $(\xi, \xi) = \sum_a \xi_a^2$. Then*

$$\mathbb{E} e^{\frac{1}{2}(\xi, \xi)} \leq e^{\sum_a C(a, a)}. \quad (6.24)$$

Proof. Let $t \in (0, 1)$. It suffices to show that

$$\frac{d}{dt} \ln \mathbb{E} e^{\frac{t}{2}(\xi, \xi)} \leq \sum_{a \in \mathcal{A}} C(a, a), \quad (6.25)$$

since the desired inequality then follows by integration over $t \in (0, 1)$.

Let A be the inverse of the matrix C . The eigenvalues of A are at least 2 by the hypothesis on C , so the inverse matrix $C_t = (A - t)^{-1}$ exists. Let \mathbb{E}_t denote the Gaussian expectation with covariance C_t . Then

$$\frac{d}{dt} \ln \mathbb{E} e^{\frac{t}{2}(\xi, \xi)} = \frac{1}{2} \mathbb{E}_t(\xi, \xi) = \frac{1}{2} \sum_{a \in \mathcal{A}} C_t(a, a) = \frac{1}{2} \text{Trace } C_t = \frac{1}{2} \sum_{\lambda} (\lambda^{-1} - t)^{-1}, \quad (6.26)$$

where the sum over λ runs over the eigenvalues of C (with multiplicity). Since each λ is at most $\frac{1}{2}$ by hypothesis, $(\lambda^{-1} - t)^{-1} = \lambda(1 - t\lambda)^{-1} \leq 2\lambda$, and hence

$$\frac{d}{dt} \ln \mathbb{E} e^{\frac{t}{2}(\xi, \xi)} \leq \sum_{\lambda} \lambda = \text{Trace } C = \sum_{a \in \mathcal{A}} C(a, a), \quad (6.27)$$

which completes the proof. ■

Lemma 6.6 (Lattice Sobolev inequality). *Let $f : B \rightarrow \mathbb{C}$, where $B \in \mathcal{B}$ is a block of side length R . Let $\nabla_R = R\nabla$. Then for any $x \in B$,*

$$|f(x)|^2 \leq 2^{3d+2} R^{-d} \sum_{y \in B} \sum_{|\alpha|_{\infty} \leq 1} |\nabla_R^{\alpha} f(y)|^2. \quad (6.28)$$

Proof. We can choose coordinates on B such that $B = \{0, 1, \dots, R-1\}^d$. Let $g : B \rightarrow \mathbb{R}$ be any function that vanishes on $\cup_{i=1}^d \{(x_1, \dots, x_d) \in B : x_i = 0\}$. Then we have the telescoping sum

$$g(x) = \sum_{y: y_i < x_i \forall i} \nabla^{e_1} \dots \nabla^{e_d} g(y). \quad (6.29)$$

Therefore, by the Cauchy–Schwarz inequality,

$$|g(x)| \leq \sum_{y \in B} |\nabla^{e_1} \dots \nabla^{e_d} g(y)| \leq (|B| \sum_{y \in B} |\nabla^{e_1} \dots \nabla^{e_d} g(y)|^2)^{1/2}. \quad (6.30)$$

We apply this to $g(x) = x_1 \cdots x_d f(x)$, for points $x \in B$ with each coordinate $x_i \geq R/2$. This gives

$$|f(x)| \leq \left(\frac{2}{R}\right)^d |x_1 \cdots x_d f(x)| \leq 2^d (|B|^{-1} \sum_{y \in B} |\nabla^{e_1} \cdots \nabla^{e_d} y_1 \cdots y_d f(y)|^2)^{1/2}. \quad (6.31)$$

We evaluate the derivatives using $\nabla^{e_i} y_i h(y) = y_i \nabla^{e_i} h(y) + \nabla^{e_i} h(y) + h(y)$. Since $y_i \leq R$,

$$|f(x)|^2 \leq 2^{2d} |B|^{-1} \sum_{y \in B} \left(\sum_{\alpha \in \{0,1\}^d} 2 |\nabla_R^\alpha f(y)| \right)^2 \leq 2^{2d+2} |B|^{-1} \sum_{y \in B} 2^d \sum_{\alpha \in \{0,1\}^d} |\nabla_R^\alpha f(y)|^2. \quad (6.32)$$

Since this holds for all functions f we can change variables by reflections through hyperplanes bisecting B so as to remove the assumption that every coordinate x_i obeys $x_i \geq R/2$. These reflections turn forward derivatives into backward derivatives, and we obtain (6.28) by noticing that the absolute value of a backward derivative equals the absolute value of a forward derivative at a neighbouring point. \blacksquare

Recall the definition of $G(X, \phi)$ in Definition 3.14, for $X \in \mathcal{P}$ a polymer as in Definition 3.13.

Lemma 6.7. *For $X \subset \Lambda$, $t \geq 0$, and $\phi \in \mathbb{C}^\Lambda$,*

$$G^t(X, \phi) \leq \exp \left[\frac{1}{2} \sum_{y \in X^\square} \sum_{|\alpha|_1 \leq d+p_\Phi} |\xi(y, \alpha)|^2 \right], \quad (6.33)$$

where $\xi(y, \alpha) = ct^{1/2} R^{-d/2} \ell^{-1} \nabla_R^\alpha \phi(y)$ for some constant c depending only on d .

Proof. By definition,

$$G^t(X, \phi) = \exp \left[t \sum_{x \in X} |B_x|^{-1} \|\phi\|_{\Phi(B_x^\square, \ell)}^2 \right], \quad (6.34)$$

so it suffices to show that

$$t \sum_{x \in X} |B_x|^{-1} \|\phi\|_{\Phi(B_x^\square, \ell)}^2 \leq \frac{1}{2} \sum_{y \in X^\square} \sum_{|\alpha|_1 \leq d+p_\Phi} |\xi(y, \alpha)|^2. \quad (6.35)$$

Throughout the proof, c denotes a d -dependent constant whose value may change from line to line. Note that for $B \in \mathcal{B}$, B^\square is a cube (since connectivity of blocks can be via corners) whose side length is a d -dependent multiple of R . We first apply Lemma 6.6 with $f(x) = \nabla_R^\alpha \phi(x)$ and B replaced by B^\square to obtain, for $x \in B^\square$,

$$|\nabla_R^\alpha \phi(x)|^2 \leq cR^{-d} \sum_{y \in B^\square} \sum_{|\alpha'|_\infty \leq 1} |\nabla_R^{\alpha+\alpha'} \phi(y)|^2. \quad (6.36)$$

From this, we obtain

$$\|\phi\|_{\Phi(B^\square, \ell)}^2 \leq \max_{|\alpha|_1 \leq p_\Phi, x \in B^\square} |\ell^{-1} \nabla_R^\alpha \phi(x)|^2 \leq cR^{-d} \sum_{y \in B^\square} \sum_{|\alpha|_1 \leq d+p_\Phi} |\ell^{-1} \nabla_R^\alpha \phi(y)|^2. \quad (6.37)$$

If $y \in B_x^\square$ then $x \in B_y^\square$ and $|B_y^\square|/|B|$ is bounded by a geometric constant. With a larger value of c , this gives

$$\begin{aligned} t \sum_{x \in X} |B_x|^{-1} \|\phi\|_{\Phi(B_x^\square, \ell)}^2 &\leq ctR^{-d} \sum_{y \in X^\square} \sum_{|\alpha|_1 \leq d+p_\Phi} |\ell^{-1} \nabla_R^\alpha \phi(y)|^2 \\ &= \frac{1}{2} \sum_{y \in X^\square} \sum_{|\alpha|_1 \leq d+p_\Phi} |\xi(y, \alpha)|^2, \end{aligned} \quad (6.38)$$

and the proof is complete. \blacksquare

Now we restate, and prove, Proposition 3.20 as the following lemma. Recall that the $\Phi^+(\ell)$ norm is the $\Phi(\ell)$ norm with p_Φ increased to $p_\Phi + d$.

Lemma 6.8. *Let $t \geq 0$, $\alpha_G > 1$, and let $X \subset \Lambda$. There exists a (small) positive constant $c(\alpha_G)$, which is independent of R , such that if $\|\mathbf{C}_b\|_{\Phi^+(\ell)} \leq c(\alpha_G)t^{-1}$, then*

$$0 \leq \mathbb{E}_{\mathbf{C}_b} G^t(X, \phi) \leq \alpha_G^{R^{-d}|X|}. \quad (6.39)$$

Proof. By Lemma 6.7,

$$\mathbb{E}_{\mathbf{C}_b} G^t(X, \phi) \leq \mathbb{E}_{\mathbf{C}_b} \exp \left[\frac{1}{2} \sum_{y \in X^\square} \sum_{|\alpha|_1 \leq d+p_\Phi} |\xi(y, \alpha)|^2 \right]. \quad (6.40)$$

The variables $\xi(x, \alpha)$ are Gaussian and we denote their covariance by Q . The largest eigenvalue λ_{\max} of Q is at most the norm of Q considered a convolution operator on $l^2(X^\square)$. Therefore, using Young's inequality we obtain

$$\lambda_{\max} \leq \sup_{f: \|f\|_2 \leq 1} \|Q * f\|_2 \leq \|Q\|_1 \leq cR^d \|Q\|_\infty. \quad (6.41)$$

Since Q is a positive-definite function, its maximum value occurs on the diagonal, and obeys

$$\|Q\|_\infty \leq ctR^{-d} \max_{|\alpha|_1 \leq d+p_\Phi, x \in X^\square} |\ell^{-2} \nabla_R^{2\alpha} \mathbf{C}_{b;x,x}| \leq ctR^{-d} \|\mathbf{C}_b\|_{\Phi^+}, \quad (6.42)$$

so

$$\lambda_{\max} \leq ct \|\mathbf{C}_b\|_{\Phi^+}. \quad (6.43)$$

This will be less than $\frac{1}{2}$ if $\|\mathbf{C}_b\|_{\Phi^+} \leq c(d)t^{-1}$ with $c(d)$ sufficiently small. We may therefore apply Lemma 6.5 with ξ_a replaced by $\xi(x, \alpha)$. This gives

$$\mathbb{E}_{\mathbf{C}_b} G^t(X, \phi) \leq e^{\sum_{y \in X^\square} \sum_{|\alpha|_1 \leq d+p_\Phi} \text{Var}(\xi(y, \alpha))}. \quad (6.44)$$

Since $\text{Var}(\xi(y, \alpha)) \leq ctR^{-d} \|\mathbf{C}_b\|_{\Phi^+}$ this gives

$$\mathbb{E}_{\mathbf{C}_b} e^{t \sum_{b \in \mathcal{B}_j(X)} \|\xi\|_{\Phi_j(b^\square, \ell_j)}^2} \leq e^{ct \|\mathbf{C}_b\|_{\Phi^+} R^{-d}|X^\square|}, \quad (6.45)$$

and the desired result follows since $|X^\square| \leq a|X|$ for some $a = a(d)$. \blacksquare

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